SILHOUETTE OF A RANDOM POLYTOPE*

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Abstract. We consider random polytopes defined as the convex hull of a Poisson point process on a sphere in $\mathbb{R}^3$ such that its average number of points is n. We show that the expectation over all such random polytopes of the maximum size of their silhouettes viewed from infinity is $\Theta(\sqrt{n})$.

1 Introduction

The silhouette of a polytope in $\mathbb{R}^3$ with respect to a given viewpoint at infinity is, roughly speaking, the set of edges incident to a front and a back face. The size of the silhouette is its number of edges or equivalently vertices. Silhouettes of (non-necessarily convex) polyhedra naturally arise in computer graphics for hidden surface removal or shadow computation [5, 6]. They are also important in shape recognition; Sander et al. [14] claim that the silhouette "is one of the strongest visual cues of the shape of an object".

While the silhouette size of a polyhedron with n vertices may be linear for some viewpoints, several experimental and theoretical studies showed a sublinear behavior for a wide range of constraints. The latest result on the subject proves a bound in $O(\sqrt{n})$ on the size of the silhouette from a random viewpoint of polyhedra of size n approximating non-convex surfaces in a reasonable way [8]. Prior to this result, it was widely accepted that the silhouette of a polyhedron is often of size $\Theta(\sqrt{n})$ as, for instance, stated by Sander et al. [14]. An experimental study by Kettner and Welzl [10] confirmed this for a set of realistic objects, study which was extended by McGuire [13] to a larger database of larger objects for which the observed silhouette size was approximately $n^{0.8}$. In terms of theoretical results, Kettner and Welzl [10] first proved the $\Theta(\sqrt{n})$ bound for the size of the silhouette, viewed from a random point at infinity, of a convex polyhedron that approximates a sphere with small Hausdorff distance. Alt et al. [1] also gave conditions under which the silhouette of a convex polyhedron has sub-linear size in the worst case.

This paper addresses the size of the silhouettes of a random polytope. Previous work focuses on the expected complexity of such silhouettes, averaged over all points of view. In particular, Borgwardt [2] and later Küfer [12, 11] bounded this expected complexity in any dimension, focusing on the constants. In these papers, the authors considered random polytopes defined either as the convex hull of points that are uniformly spherically symmetric distributed [12], or as the dual polytope of points that are uniformly distributed on the unit

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sphere [2, 11]. Under these distributions, their results yield that the expected complexity of the silhouettes, averaged over all points of view, is $\Theta(\sqrt{n})$ in $\mathbb{R}^3$. Küfer also addressed the question of the deviations from the mean in concluding remarks [12].

In this paper, we address the size of the silhouette of a random polytope from the *worst-possible* viewpoint at infinity. We consider a Poisson point process on a sphere so the average number of points is n and we define a random polytope as the convex hull of the Poisson point process. We do not pretend that random polytopes are a good model of the objects used in graphics, but this result gives further insight explaining why silhouettes tend to be small, and the proof techniques are interesting in their own right. Our main result is the following.

Theorem 1. *The expectation over all random polytopes of the maximum size of their silhouettes viewed from infinity is* $\Theta(\sqrt{n})$.

We first prove in Section 3 that the size of the worst-case silhouette of a random polytope viewed from infinity is in $O(\sqrt{n \ln n})$. We then refine this analysis in Section 4 and prove that this expected maximum size is in $\Theta(\sqrt{n})$.

This paper uses as a starting point the technique introduced by Devillers et al. [3]. One example they consider to illustrate their technique is the expected size of the convex hull of points sampled according to a uniform Poisson process in a disk, which they bound by $O(n^{1/3} \operatorname{polylog}(n))$ where n is the mean of the Poisson process (this is a weaker version of a well known result, but the point is the simplicity of their proof). In the beginning of Section 3, we adapt their analysis to silhouettes viewed from a fixed direction in a straightforward manner by only modifying the density of the Poisson process. We then extend their technique in two main directions. First, we prove that our upper bounds are "reliable" in the sense that the probability of a large deviation is very small, i.e., it is very unlikely that the variables get much larger than these upper bounds on their expectation. This will allow us to bound the expectation of the maximum of a number of variables and thus to bound the *worst-case* size of the silhouette of a random polytope. Second, in Section 4, we refine the analysis in order to remove the polylogarithmic factor. The techniques introduced here are fairly generic and they can be used in other problems for removing polylogarithmic factors in expected complexities, as demonstrated by Devillers et al. [4].

It should be stressed that the technique of Kettner and Welzl [10] does not easily extend to give a bound on the worst-case silhouette of random polytopes. Indeed, they compute the expected size of the silhouette as the sum of the dihedral angles of the edges and their approach is thus intrinsically tied to the average over all viewpoints.

2 Preliminaries

We define a random polytope as the convex hull of a point process on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. For a subset B of the sphere, let $\mathcal{A}(B)$ be its area. More formally, we consider a Poisson point process on $\mathbb{S}^2$ with intensity $\frac{n}{\mathcal{A}(\mathbb{S}^2)}$, so that the mean number of vertices of the polytope is n.

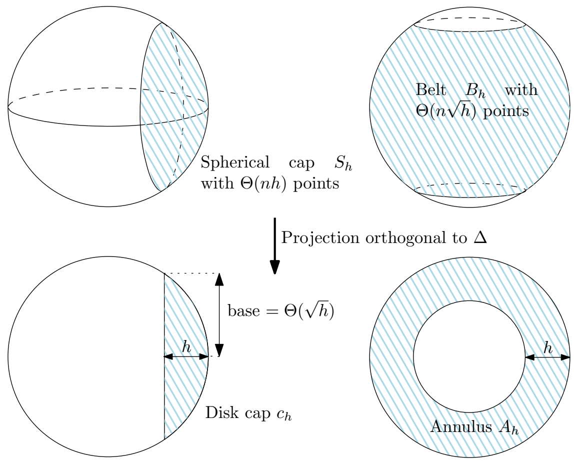

Figure 1: Left: disk and spherical caps, right: annulus and belt.

For a direction Δ that is coplanar with no face of a polytope, the silhouette of the polytope viewed from that direction is the set of its edges that are adjacent to a front face and a back face, where front and back faces are defined by the sign of the scalar product of their outer normal with Δ, and the size of the silhouette is the number of its edges or equivalently vertices. The case where Δ is coplanar with a face arises with probability zero but the silhouette size has to be defined since we consider the worst-case size for all viewing directions. Various definitions for the silhouette are possible (see for instance [8]) but, instead of detailing the possible definitions, we simply bound the size of the silhouette by the number of polytope vertices that are incident to two faces whose outer normals have scalar products with Δ whose product is non-positive.

Next, we state some preliminary results that could be skimmed at first and referred to when needed.

Poisson distribution. For a subset B of the sphere, denote by $N(B)$ the number of points of the Poisson point process that fall in B. The random variable $N(B)$ follows a Poisson distribution of parameter $n\frac{\mathcal{A}(B)}{\mathcal{A}(\mathbb{S}^2)}$ so that

- for any $k \in \mathbb{N}$, the probability that $N(B) = k$, denoted $\mathbf{P}\left(N(B) = k\right)$ is equal to $\frac{\left(n\frac{\mathcal{A}(B)}{\mathcal{A}(\mathbb{S}^2)}\right)^k}{k!} e^{-n\frac{\mathcal{A}(B)}{\mathcal{A}(\mathbb{S}^2)}}$, and
- the expectation of $N(B)$, denoted $\mathbf{E}\left[N(B)\right]$, is $n\frac{\mathcal{A}(B)}{\mathcal{A}(\mathbb{S}^2)}$.

Spherical geometry. In a plane orthogonal to a direction Δ, we denote by D the projection of the sphere $\mathbb{S}^2$ (see Figure 1). Assume that D is centered at the origin of the Euclidean plane with a coordinate system (x, y). We define the disk cap c_h as the set of points in the disk D with $x \geqslant 1-h$, we call h its height and define its base as its maximum y-coordinate. We denote S_h the spherical cap of $\mathbb{S}^2$ that projects to the disk cap c_h. Elementary geometric calculus shows that, for all h in $[0, 2]$, the base is in $\Theta(\sqrt{h})$, the area of S_h is in $\Theta(h)$ and thus the expected number of points in this cap is in $\Theta(nh)$.

In addition, we define A_h the annulus of width h as the subset of D bounded by the circles of radii $(1-h)$ and 1. We also define the belt B_h as the part of the sphere that projects to the annulus A_h. Elementary geometric calculus shows that the width h of an annulus and the area $\mathcal{A}(B_h)$ of the belt B_h satisfy, for all h in $[0,1]$, the relations $\mathcal{A}(B_h)=\Theta(\sqrt{h})$ and $h=\Theta(\mathcal{A}(B_h)^2)$. Hence, the expected number of points in the belt B_h is $\mathbf{E}\left[N(B_h)\right]=\Theta(n\sqrt{h})$.

Large deviations. The following lemma states a large deviation principle for a Poisson distribution. It is a classical simple result, a detailed proof can be found for instance in the appendix of [7].

Lemma 2. *Let N_λ be a random variable following a Poisson distribution of parameter λ, then for any $\eta>0$,*

$$\mathbf{P}\left(N_\lambda \geqslant \lambda(1+\eta)\right) \leqslant e^{-\lambda I(\eta)}$$

with $I(\eta)=(1+\eta)\ln(1+\eta)-\eta$; note that $I(\eta)>0$ for $\eta>0$.

A refinement of Hoeffding's inequality that allows some dependency between the variables is the following. Let $(Y_1,\ldots,Y_m)$ be random variables not necessarily independent. A family $\{A_j\}_j$ of subsets of indices in $\{1,\ldots,m\}$ is a cover if $\cup_j A_j=\{1,\ldots,m\}$. The cover is proper if, in each subset A_j, the random variables Y_i are independent. We denote by χ the size of a smallest proper cover, i.e., the smallest k such that $\{1,\ldots,m\}$ is the union of k subsets of indices of independent variables.

Lemma 3. *[9, Theorem 2.1] Let $(Y_1,\ldots,Y_m)$ be random variables with range $[a_i,b_i]$ and $S_m=Y_1+\cdots+Y_m$, then for any $t>0$*

$$\mathbf{P}\left(S_m-\mathbf{E}\left[S_m\right]\geqslant t\right)\leqslant \exp\left(-2\frac{t^2}{\chi\sum_{i=1}^m(b_i-a_i)^2}\right).$$

3 Simple proof yielding a $O(\sqrt{n\ln n})$ bound

We prove in this section the following proposition which says, roughly speaking, that the size of the worst-case silhouette of a random polytope viewed from infinity is in $O(\sqrt{n\ln n})$.

Proposition 4. *Consider a Poisson point process on $\mathbb{S}^2$ with intensity $\frac{n}{\mathcal{A}(\mathbb{S}^2)}$, and the polytope defined as the convex hull of the points. The expectation over all such random polytopes of the maximum size of their silhouettes viewed from infinity is $O(\sqrt{n\log n})$.*

We consider points of view at infinity, thus the silhouette is the convex hull of the projection of the points on a plane orthogonal to the viewpoint direction. For a viewpoint direction Δ, we denote by $\mathrm{silh}(\Delta)$ the number of vertices of the silhouette viewed from Δ. We successively study the size of the silhouette from a fixed viewpoint, in a neighborhood of a fixed view point, and then consider all possible viewpoints. The idea is that the projection of the silhouette vertices are expected to be in an annulus of small width, or equivalently that the silhouette vertices are expected to be in a belt of small width on the sphere. Then, for nearby viewpoints, we show that all possible silhouette points are expected to remain

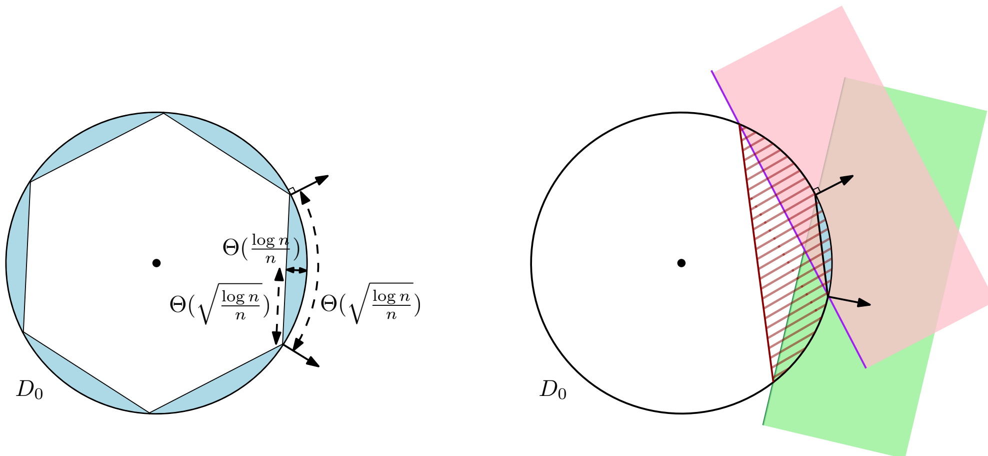

Figure 2: (a) $\Theta(\sqrt{\frac{n}{\ln n}})$ witnesses in the projected sphere D_0. (b) Collector (hatched) of a witness (in blue).

in a slightly larger belt. Finally, a covering argument of the set of viewpoints at infinity by such neighborhoods and a large deviation principle enable us to conclude.

Silhouette from a fixed viewpoint at infinity. The silhouette of a polytope viewed from a fixed direction Δ_0 is, in projection onto a plane orthogonal to Δ_0, the convex hull of the projected polytope vertices. We are thus analyzing the size of the convex hull of points distributed in a disk according to the (non-homogeneous) point process induced by the Poisson point process of the points on the sphere.

We consider the circle C_0 bounding the disk D_0 that is the projection of the sphere $\mathbb{S}^2$ in the direction orthogonal to Δ_0. We partition C_0 into arcs and consider the convex hulls of each of these arcs, which we call *witnesses*, and the spherical witnesses caps of $\mathbb{S}^2$ that project on these witnesses (see Figure 2(a)). Associated to every witness, we consider the *range* of directions (in the plane orthogonal to Δ_0) defined by the rays starting at the center of disk D_0 and that intersect the witness, and we define the *collector* associated to every witness as (see Figure 2(b)) the convex hull[1] of the union (clipped by disk D_0) of the half-planes whose inward normals are in the corresponding range and whose boundaries intersect the witness.

The **key property of witnesses, ranges and collectors** is that if a witness contains a point, any point that is extreme in one of the directions of the associated range has to belong to the associated collector; furthermore the height of the collector is larger than that of its witness by at most a constant multiplicative factor (since the circular arc defining the collector is exactly three times longer than the one defining the witness). Hence, if none of the witnesses is empty, the projected vertices of the silhouette are contained in the union of the collectors. In this case, denoting h the height of the witnesses, the union of the collectors is included in an annulus of width $\Theta(h)$. Using the spherical geometric

[1] Considering the convex hull is not necessary but convenient since both the witness and the collector have the geometry of a disk cap as defined in the preliminaries.

preliminaries, the vertices of the silhouette are thus in a belt of area $\Theta(\sqrt{h})$. Conversely, for the belt to contain an expected number of $\Theta(f(n))$ points, its area should be in $\Theta(f(n)/n)$ and the height of the witnesses should be $h = \Theta((f(n)/n)^2)$.

To bound the expected size of the silhouette from the viewpoint Δ_0, it is then sufficient to select an adequate value for the height of the witnesses such that: (a) when none of the witnesses is empty, the belt containing the silhouette has the expected number of points stated in Proposition 4, $O(\sqrt{n \log n})$, and (b) the probability that at least one of the witnesses is empty is small (e.g. is in $O(\frac{1}{n})$, in which case the conditional expected silhouette can be crudely bounded by n).

Note that this setting fails to prove the upper bound of Theorem 1. Indeed, if one wants the expected number of vertices in the belt to be in $\Theta(\sqrt{n})$, this implies that the height of the witnesses is $h = \Theta((\frac{\sqrt{n}}{n})^2) = \Theta(\frac{1}{n})$. The probability that a given witness is empty is then a constant (since the law of the number of points in the witness is a Poisson distribution of parameter $\Theta(1)$) and the probability that at least one witness is empty is not asymptotically small, thus property (b) is not satisfied.

To prove Proposition 4, we take a larger belt that contains an expected number of vertices in $\Theta(\sqrt{n \ln n})$, which implies that the height of the witnesses is $\Theta(\frac{\ln n}{n})$ and that the expected number of vertices in each witness cap is $\Theta(\ln n)$. Precisely, we set the size of the ranges and witnesses so that the expected number of vertices in each witness cap is $\alpha \ln n$ where α is a constant which we define later. The base of a witness is thus in $\Theta(\sqrt{\frac{\alpha \ln n}{n}})$. It follows that the length of the circular arc defining a witness is also in $\Theta(\sqrt{\frac{\alpha \ln n}{n}})$ and thus that the number of witnesses is in $\Theta(\sqrt{\frac{n}{\alpha \ln n}})$.[2] Property (b) is then satisfied since the probability that a given witness is empty is small; indeed (see Section 2), since the expected number of points in the witness cap is $\alpha \ln n$, the corresponding Poisson distribution has parameter $\alpha \ln n$ and thus the probability that there is no point in the cap is $e^{-\alpha \ln n} = n^{-\alpha}$. The probability that (at least) one of the $\Theta(\sqrt{\frac{n}{\alpha \ln n}})$ witnesses is empty is thus at most $O(\sqrt{\frac{n}{\alpha \ln n}}) \cdot n^{-\alpha}$ and thus is $O(n^{\frac{1}{2}-\alpha})$.

We conclude by computing the expectation of the size of the silhouette conditioned by the events that all the witnesses are empty or not:

$$\begin{aligned}\mathbf{E}\left[\operatorname{silh}(\Delta_0)\right] =& \mathbf{E}\left[\operatorname{silh}(\Delta_0) | \exists i, N(W_i) = 0\right] \mathbf{P}\left(\exists i, N(W_i) = 0\right) \\ &+ \mathbf{E}\left[\operatorname{silh}(\Delta_0) | \forall i, N(W_i) \neq 0\right] \mathbf{P}\left(\forall i, N(W_i) \neq 0\right).\end{aligned}$$

For the first term, i.e., if (at least) one witness is empty, we bound the expected size of the silhouette by the expected number of points on the sphere, which is at most n since knowing that at least one witness is empty can only decrease the expected number of points on the sphere:

$$\mathbf{E}\left[\operatorname{silh}(\Delta_0) | \exists i, N(W_i) = 0\right] \leqslant \mathbf{E}\left[N(\mathbb{S}^2) | \exists i, N(W_i) = 0\right] \leqslant \mathbf{E}\left[N(\mathbb{S}^2)\right] = n$$

On the other hand, as noted above, if none of the witnesses is empty, the vertices of the silhouette are contained in the belt defined by the witness caps which were defined so that

[2] The constant α fixes the length of the circular arc defining the witnesses. For this length to divide the circumference of C_0, α must satisfy a constraint but this technical detail is not relevant in the proof.

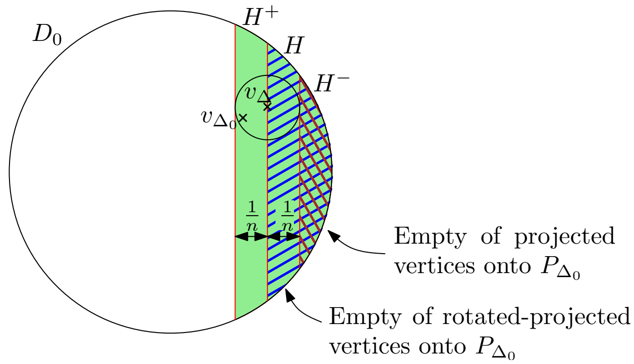

Figure 3: Empty half-planes (clipped by D_0) associated to silhouettes from a neighborhood of a fixed viewpoint.

the belt contains an expected number of vertices in $\Theta(\sqrt{n \ln n})$. Denoting by Belt_0 this belt and by $\mathbf{1}$ the characteristic function, we get:[3]

$$\begin{aligned}\mathbf{E}\left[\text{silh}(\Delta_0)|\forall i, N(W_i) \neq 0\right]\mathbf{P}\left(\forall i, N(W_i) \neq 0\right) &= \mathbf{E}\left[\text{silh}(\Delta_0) \times \mathbf{1}_{\{\forall i,\, N(W_i)\neq 0\}}\right] \\ &\leqslant \mathbf{E}\left[N(\text{Belt}_0) \times \mathbf{1}_{\{\forall i,\, N(W_i)\neq 0\}}\right] \\ &\leqslant \mathbf{E}\left[N(\text{Belt}_0)\right] = O(\sqrt{n \ln n}).\end{aligned}$$

Summing the two terms thus gives $\mathbf{E}\left[\text{silh}(\Delta_0)\right] \leqslant n\, O(n^{\frac{1}{2}-\alpha}) + O(\sqrt{n \ln n})$ which is in $O(\sqrt{n \ln n})$ by choosing $\alpha \geqslant 1$.

Silhouettes from a neighborhood of a fixed viewpoint at infinity. We now consider the expected size of the worst-case silhouette for a set of directions in some neighborhood of Δ_0, that is the set of directions that make an angle at most $\frac{1}{n}$ with Δ_0. We denote by $\Delta \sim \Delta_0$ a direction in this neighborhood and we are thus interested in the variable $\max_{\Delta\sim\Delta_0} \text{silh}(\Delta)$. As before, we consider the witnesses and collectors in a plane P_{Δ_0} orthogonal to Δ_0. Also as before, if (at least) one witness is empty, we bound the expected size of the worst-case silhouette by n which is the expected total number of points. We now assume that no witness is empty in the plane P_{Δ_0}.

The idea is to show that if a point is on the silhouette for the direction Δ, then its projection in the direction Δ_0 (onto P_{Δ_0}) is, roughly speaking, in a collector enlarged by $\frac{2}{n}$.

Consider the silhouette of the polytope in a direction Δ. We rotate the polytope (about the center of $\mathbb{S}^2$) so that direction Δ is mapped to Δ_0. Then, we project the vertices of the rotated polytope onto P_{Δ_0}. We now have two sets of points on P_{Δ_0}: the projection of the vertices of the polytope and the projection of the vertices of the rotated polytope, and there is a trivial one-to-one mapping between the points in each set. We denote by v_Δ the image of a vertex v by this rotation and projection onto P_{Δ_0}, while we refer to v_{Δ_0} as

[3]The first equality is a classical probability property which can be proved as follows with the obvious change of notation: $\mathbf{E}\left[A|B\right]\mathbf{P}\left(B\right) = \sum_k k\,\mathbf{P}\left(A=k|B\right)\ \mathbf{P}\left(B\right) = \sum_k k\,\mathbf{P}\left(A=k\ \&\ B\right) = \sum_k k\,\mathbf{P}\left(A \times \mathbf{1}_B = k\right) = \mathbf{E}\left[A \times \mathbf{1}_B\right]$.

the projection of v onto P_{Δ_0}. Note that for any vertex v, the distance between v_{Δ_0} and v_Δ is at most $\frac{1}{n}$ since the vertices of the polytope move by distance at most $\frac{1}{n}$ through the rotation and the projection onto P_{Δ_0} decreases relative distances. In the following, we refer (in two or three dimensions) to a half-space through a point v as to an open half-space whose boundary contains v.

If a vertex v is on the silhouette for the direction Δ, there is a half-space through v and parallel to Δ that is empty of vertices. Through the rotation and projection onto P_{Δ_0}, this gives a half-plane through the image v_Δ of v that is empty of the rotated and projected vertices (v'_Δ) of the polytope. Denote this half-plane by H and consider, in P_{Δ_0}, the two orthogonal translations of this half-plane by distance $\frac{1}{n}$; denote them by $H^\pm$ such that $H^- \subset H \subset H^+$ and refer to Figure 3. Since the distance between two points that are in correspondence through the one-to-one mapping is at most $\frac{1}{n}$, the half-plane H^- contains no projection of the vertices of the (non-rotated) polytope. Thus, the part of H^- that lies in the disk D_0 belongs to a collector, by definition since we assumed that no witness is empty.

As argued above, a collector has height $\Theta(\frac{\ln n}{n})$, thus $H^- \cap D_0$ is included in an annulus of width $\Theta(\frac{\ln n}{n})$ in D_0. It follows that $H^+ \cap D_0$ is included in an annulus of width $\Theta(\frac{\ln n}{n}) + \frac{2}{n} = \Theta(\frac{\ln n}{n})$.[4] Finally, observe that the projection v_{Δ_0} of v lies in $H^+ \cap D_0$ since it is at distance at most $\frac{1}{n}$ from v_Δ which lies on the boundary of H. Thus all the vertices that appear on a silhouette for a direction in the neighborhood of Δ_0 lie, in projection on P_{Δ_0}, in an annulus of width $\Theta(\frac{\ln n}{n})$, and thus lie on the unit sphere in a belt with an expected number of points in $O(\sqrt{n \log n})$.

We conclude again by computing the conditional expectation. The expected size of the worst-case silhouette for the directions in the neighborhood of Δ_0 is at most n times the probability that (at least) a witness is empty which is in $O(n^{\frac{1}{2}-\alpha})$, plus the expected number of vertices in a belt which is in $O(\sqrt{n \log n})$. The expected size is thus $\mathbf{E}\left[\max_{\Delta \sim \Delta_0} \mathrm{silh}(\Delta)\right] = O(\sqrt{n \log n})$ by choosing $\alpha \geqslant 1$.

Worst-case silhouette from any viewpoint at infinity. We cover the set of viewpoints with $O(n^2)$ disks centered on directions Δ_i. For each such disk, the expected maximum of the size of the silhouettes is $O(\sqrt{n \log n})$. Using a large deviation technique, we show that there is a low probability that the silhouette from Δ_0 (or any Δ_i) exceeds its expectation by much.

First, by setting $\alpha \geqslant 3$, we ensure that the probability that there is a Δ_i with an empty witness is at most $O(n^2\, n^{1/2-\alpha}) = O(1/\sqrt{n})$ and thus the contribution of this case to the expected maximum is $O(\sqrt{n})$.

Second, we now assume that no Δ_i has an empty witness. Then, we consider the number of points that fall within the belt associated with the neighborhood of a direction Δ_i. It follows a Poisson distribution of parameter $\beta\sqrt{n \ln n}$, thus according to Lemma 2, the probability that this number is larger than $2\beta\sqrt{n \ln n}$ is $O(e^{-\gamma\sqrt{n \ln n}})$ with $\gamma > 0$. This means that the probability that at least one of the belts associated with the Δ_i contains more

[4] Note that it is not correct that $H^+ \cap D_0$ necessarily belongs to the collector enlarged by a translation of $\frac{2}{n}$ because the boundary (in D_0) of H^+ and of the collector are not parallel. However, it is included in the enlarged annulus.

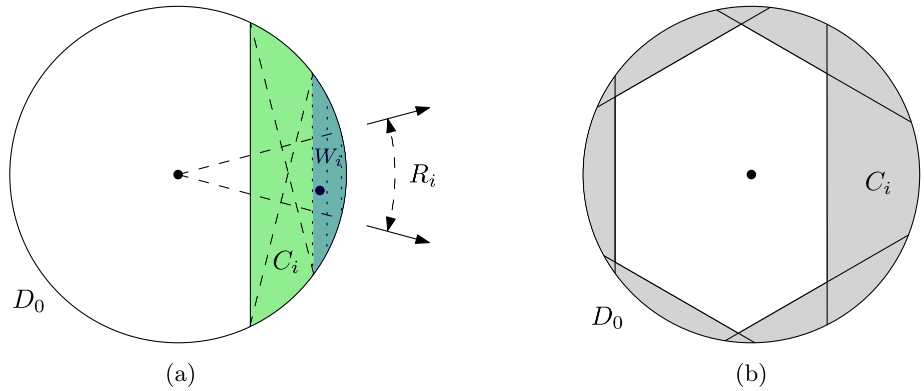

Figure 4: (a) New witness and collector in the projected sphere D_0. (b) Non-uniform annulus formed by the union of collectors.

than $2\beta\sqrt{n \ln n}$ is $O(n^2 e^{-\gamma\sqrt{n \ln n}})$, and thus the contribution of that case to the expected maximum is $O(n^3 e^{-\gamma\sqrt{n \ln n}})$.[5] The only remaining case is when all belts contain at most $2\beta\sqrt{n \ln n}$ points, in which case the maximum is $O(\sqrt{n \ln n})$.

Summing over all cases, we obtain that the expected maximum size of the silhouette is $O(\sqrt{n \ln n})$.

4 Refined proof yielding a $\Theta(\sqrt{n})$ bound

We prove in this section Theorem 1 which states that the expected size of the worst-case silhouette of a random polytope viewed from infinity is in $\Theta(\sqrt{n})$. We first prove the upper bound. We follow the same proof strategy as in Section 3, except that we consider witnesses and collectors of variable sizes.

Silhouette from a fixed viewpoint at infinity. For defining the witnesses, we first partition the circle in $\lfloor\sqrt{n}\rfloor$ ranges R_i (instead of $\Theta(\sqrt{\frac{n}{\ln n}})$ as previously), so that the associated caps of D_0 then have height $\Theta(\frac{1}{n})$. Since many such caps will be empty with high probability, we do not use them directly as witnesses (indeed we have seen that the method of Section 3 fails in this case). Instead we define, for each range R_i, the witness W_i with height $\frac{d_i}{n}$ where $d_i \in \{1, \ldots, 2n\}$ is the smallest integer such that W_i is non-empty (see Figure 4(a)). In the rare case where the whole sphere contains no point at all, we define the witnesses W_i as $\mathbb{S}^2$ and set $d_i = 2n$. Note that the witnesses will overlap with high probability.

[5] The contribution of that case to the expected maximum is $O(n^3 e^{-\gamma\sqrt{n \ln n}})$ because it is at most the product of $O(n)$, the expected number of points on the sphere, and $O(n^2 e^{-\gamma\sqrt{n \ln n}})$, the probability that at least one of the belts associated with the Δ_i contains more than $2\beta\sqrt{n \ln n}$. However, note that expected total number of points on the sphere knowing that one belt contains at least $2\beta\sqrt{n \ln n}$ is not n but it is less than $n + 2\beta\sqrt{n \ln n}$ and thus is in $O(n)$.

In the previous section, we were in some sense defining the size of all witnesses and collectors according to the largest d_i, thus building a "uniform" annulus. We are now adapting the size of each witness and collector to the local distribution of points. The resulting (topological) annulus is not uniform (see Figure 4(b)), but it is much smaller. Note that the d_i (and W_i) are dependent random variables.

We still define the collector C_i as the convex hull of the union (clipped by disk D_0) of the half-planes whose inward normals are in the corresponding range and whose boundaries intersect the witness.[6] Remember that the key property of collectors is that any vertex that is extreme in one of the directions of a range has to belong to the associated collector. As noticed in the previous section, the height of a collector is larger than that of its witness by at most a constant multiplicative factor. The height of collector C_i is then $\Theta(\frac{d_i}{n})$, and the area of the spherical cap $\overline{C_i}$ on the sphere that projects on C_i is also $\Theta(\frac{d_i}{n})$. This implies that the expected number of points in the collector C_i such that $d_i = k$ is $\mathbf{E}\left[N(\overline{C_i})|d_i = k\right] = O(k)$.

We can furthermore bound the expected number of points in the collector C_i without any knowledge on d_i. For $k \in \{1, \ldots, 2n\}$, $\mathbf{P}\left(d_i > k\right)$ is the probability that the candidate witness at height $\frac{k}{n}$ is empty, that is $O(e^{-\lambda k})$ for some $\lambda > 0$. The expected value of d_i is thus $\mathbf{E}\left[d_i\right] = \sum_{0<k\leqslant 2n} k \cdot \mathbf{P}\left(d_i = k\right) = \sum_{0\leqslant k\leqslant 2n} \mathbf{P}\left(d_i > k\right) = O(\sum_{0\leqslant k\leqslant 2n} e^{-\lambda\sqrt{k}}) = O(1)$. This yields that $\mathbf{E}\left[N(\overline{C_i})\right]$ is

$$\sum_k \mathbf{E}\left[N(\overline{C_i})|d_i = k\right] \mathbf{P}\left(d_i = k\right) \leqslant \sum_k O(k)\mathbf{P}\left(d_i = k\right) = O(\mathbf{E}\left[d_i\right]) = O(1).$$

We can now compute the expected size of the silhouette. First remember that every point on the silhouette is in a collector: $\mathrm{silh}(\Delta_0) \leqslant N(\cup_i \overline{C_i}) \leqslant \sum_i N(\overline{C_i})$, thus $\mathbf{E}\left[\mathrm{silh}(\Delta_0)\right] \leqslant \sum_i \mathbf{E}\left[N(\overline{C_i})\right] = O(\sqrt{n})$. For a fixed viewpoint direction Δ_0, the expected size of the silhouette is thus in $O(\sqrt{n})$.

Silhouettes from a neighborhood of a fixed viewpoint at infinity. The proof starts exactly as the one in Section 3 up to the proof that $H^- \cap D_0$ belongs to a collector C_i. Then, we argue in Section 3 that $H^+ \cap D_0$ is included in an annulus of width $\frac{2}{n}$ plus the height of the collectors. Here, since the collectors do not all have the same height, we prove instead that $H^+ \cap D_0$ belongs to a collector C'_i obtained from C_i by increasing its height by $O(\frac{1}{n})$ (see Figure 5). Postponing this proof, this yields that the height of the enlarged collector C'_i is in $O(\frac{d_i}{n})$ as for C_i. It then follows, as above, that the expected number of points in an enlarged collector remains constant and thus the maximum silhouette over a small cone of viewpoint directions with radius $\frac{1}{n}$ around Δ_0 is still in $O(\sqrt{n})$.

We now prove that $H^+ \cap D_0$ belongs to a collector C'_i obtained from C_i by increasing its height by $O(\frac{1}{n})$. Refer to Figure 5. Recall that C_i is defined as the convex hull of the union (clipped by disk D_0) of the half-planes whose inward normals are in the corresponding range and whose boundaries intersect the corresponding witness. We define similarly the enlarged collector C'_i by considering instead the half-planes translated by $\frac{2}{n}$ in the direction of their outward normal. Observe that $H^+ \cap D_0$ belongs to C'_i by construction. We now prove that the height of C'_i is that of C_i plus $O(\frac{1}{n})$.

[6] In the special case where the sphere contains no point, $C_i = W_i = \mathbb{S}^2$ and the collectors contain all the points, i.e., none.

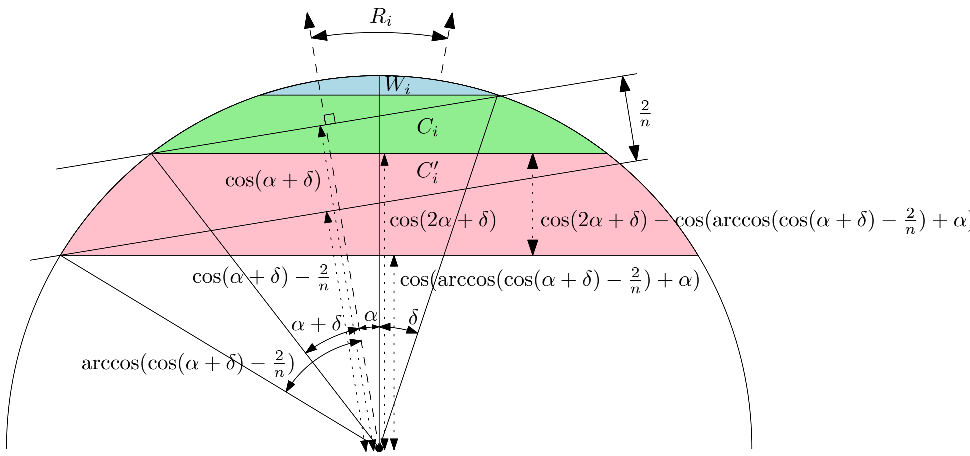

Figure 5: Enlarged collector C'_i.

Standard trigonometry (refer to Figure 5) yields that the difference of heights of C'_i and C_i is $\cos(2\alpha+\delta)-\cos(\alpha+\arccos(\cos(\alpha+\delta)-\frac{2}{n}))$ where 2α is the angle of range R_i and 2δ is the angle of the arc corresponding to the witness W_i. A Taylor expansion of that expression gives that it is in $O(\frac{1}{n})$. This concludes the proof that the maximum silhouette over a small cone of viewpoint directions with radius $\frac{1}{n}$ around Δ_0 is in $O(\sqrt{n})$.

Worst-case silhouette from any viewpoint at infinity. As in Section 3, we cover the set of viewpoints with $O(n^2)$ disks centered on directions Δ_i. To compute the worst-case silhouette over all viewpoints, we again use a large deviation technique to show that there is a low probability that the silhouette from Δ_0 (or any Δ_i) exceeds its expectation by much.

First, we reduce the problem of bounding the expectation of the maximum silhouette size to that of proving a large deviation result. For a positive constant δ to be fixed later, we consider $\delta\sqrt{n}$ as a threshold for the maximum silhouette size:

$$\begin{aligned}\mathbf{E}\left[\max_\Delta \mathrm{silh}(\Delta)\right] &= \mathbf{E}\left[\max_\Delta \mathrm{silh}(\Delta)\times \mathbf{1}_{\max_\Delta \mathrm{silh}(\Delta)\geqslant\delta\sqrt{n}}\right]\\ &\quad+\mathbf{E}\left[\max_\Delta \mathrm{silh}(\Delta)\times \mathbf{1}_{\max_\Delta \mathrm{silh}(\Delta)<\delta\sqrt{n}}\right].\end{aligned} \tag{1}$$

The second term is bounded by $\delta\sqrt{n}$. For the first one, Cauchy-Schwartz inequality yields:

$$\begin{aligned}\mathbf{E}\left[\max_\Delta \mathrm{silh}(\Delta)\times \mathbf{1}_{\max_\Delta \mathrm{silh}(\Delta)\geqslant\delta\sqrt{n}}\right] &\leqslant \mathbf{E}\left[\max_\Delta \mathrm{silh}(\Delta)^2\right]^{1/2}\mathbf{E}\left[\mathbf{1}^2_{\max_\Delta \mathrm{silh}(\Delta)\geqslant\delta\sqrt{n}}\right]^{1/2}\\ &\leqslant (n+n^2)^{1/2}\,\mathbf{P}\left(\max_\Delta \mathrm{silh}(\Delta)\geqslant\delta\sqrt{n}\right)^{1/2}\end{aligned}$$

since $N(\mathbb{S}^2)$ has a Poisson distribution of parameter n and $\mathbf{E}\left[\max_\Delta \mathrm{silh}(\Delta)^2\right]\leqslant\mathbf{E}\left[N(\mathbb{S}^2)^2\right]=\mathrm{var}(N(\mathbb{S}^2))+\mathbf{E}\left[N(\mathbb{S}^2)\right]^2=n+n^2$.

In addition, using the covering of the viewpoints by the $O(n^2)$ disks and denoting $\{\Delta\sim\Delta_0\}$ the disk of viewpoints in the neighborhood of Δ_0, one has $\mathbf{P}\left(\max_\Delta \mathrm{silh}(\Delta)\geqslant\delta\sqrt{n}\right)\leqslant O(n^2)\,\mathbf{P}\left(\max_{\Delta\sim\Delta_0}\mathrm{silh}(\Delta_0)\geqslant\delta\sqrt{n}\right)$. All that remains is to prove that $\mathbf{P}\left(\max_{\Delta\sim\Delta_0}\mathrm{silh}(\Delta_0)\geqslant\delta\sqrt{n}\right)$ is small.

Let $\mathcal{Q}$ be the event that all d_i are smaller than n^ε and all $N(\overline{C_i})$ are smaller than $2c_0n^\varepsilon$, for some ε positive to be fixed later and a constant c_0 such that $\mathbf{E}\left[N(\overline{C_i})|d_i = k\right] \leqslant c_0k$ for all k. We now distinguish two cases depending on $\mathcal{Q}$:

$$\begin{aligned}\mathbf{P}\left(\max_{\Delta\sim\Delta_0}\mathrm{silh}(\Delta_0) \geqslant \delta\sqrt{n}\right) = \mathbf{P}\left(\max_{\Delta\sim\Delta_0}\mathrm{silh}(\Delta_0) \geqslant \delta\sqrt{n} \text{ and } \mathcal{Q}\right)\\ + \mathbf{P}\left(\max_{\Delta\sim\Delta_0}\mathrm{silh}(\Delta_0) \geqslant \delta\sqrt{n} \text{ and not } \mathcal{Q}\right).\end{aligned} \tag{2}$$

The second term is bounded by (for any i)

$$\mathbf{P}\left(\text{not } \mathcal{Q}\right) \leqslant \sqrt{n}\left(\mathbf{P}\left(d_i > n^\varepsilon\right) + \mathbf{P}\left(N(\overline{C_i}) > 2c_0n^\varepsilon \mid d_i \leqslant n^\varepsilon\right)\right).$$

Remember that $\mathbf{P}\left(d_i > n^\varepsilon\right) \leqslant e^{-\lambda n^\varepsilon}$. Since for all k $\mathbf{E}\left[N(\overline{C_i})\,|\,d_i = k\right] \leqslant c_0k$, one has $\mathbf{E}\left[N(\overline{C_i})\,|\,d_i \leqslant k\right] \leqslant c_0k$ and Lemma 2 implies[7] that $\mathbf{P}\left(N(\overline{C_i}) > 2c_0n^\varepsilon \mid d_i \leqslant n^\varepsilon\right) \leqslant e^{-\Omega(n^\varepsilon)}$. Thus, $\mathbf{P}\left(\text{not } \mathcal{Q}\right)$ is exponentially small, i.e., smaller than $\sqrt{n}e^{-\Omega(n^\varepsilon)}$.

For the first term of (the right-hand side of) Eq. (2), we use the fact that the silhouette is included in the collectors: $\max_{\Delta\sim\Delta_0}\mathrm{silh}(\Delta_0) \leqslant N(\cup\overline{C_i}) \leqslant \sum N(\overline{C_i})$. In addition, we have shown that $\mathbf{E}\left[N(\overline{C_i})\right] = O(1)$ and this remains true conditionally to $\mathcal{Q}$ since this event has a probability close to 1: $\mathbf{E}\left[N(\overline{C_i})|\mathcal{Q}\right] \leqslant \mathbf{E}\left[N(\overline{C_i})\right]/\mathbf{P}\left(\mathcal{Q}\right) = O(1)$. Denoting N_i' the random variable $N(\overline{C_i})|\mathcal{Q}$, we have $\mathbf{E}\left[\sum N_i'\right] = O(\sqrt{n})$ and let c_1 be a positive constant such that $\mathbf{E}\left[\sum N_i'\right] \leqslant c_1\sqrt{n}$. Note also that the variables N_i' are bounded by $O(n^\varepsilon)$.

$$\begin{aligned}\mathbf{P}\left(\max_{\Delta\sim\Delta_0}\mathrm{silh}(\Delta_0) \geqslant \delta\sqrt{n} \text{ and } \mathcal{Q}\right) &= \mathbf{P}\left(\max_{\Delta\sim\Delta_0}\mathrm{silh}(\Delta_0) \geqslant \delta\sqrt{n} \mid \mathcal{Q}\right)\mathbf{P}\left(\mathcal{Q}\right)\\ &\leqslant \mathbf{P}\left(\sum N_i' \geqslant \delta\sqrt{n}\right)\cdot 1\\ &\leqslant \mathbf{P}\left(\sum N_i' - \mathbf{E}\left[\sum N_i'\right] \geqslant c_2\sqrt{n}\right)\end{aligned}$$

with $c_2 = \delta - c_1$ that can be chosen positive for δ large enough.

The random variables N_i' are dependent, since the witnesses are not disjoint sets. On the other hand, the fact that the d_i are bounded implies that we can find a partition of the N_i' such that, in each subset, the N_i' are independent (i.e., the $\overline{C_i}$ do not overlap) and such that the number of subsets is small. Then the extension of Hoeffding's inequality stated in Lemma 3 applies. Specifically, we will show that the smallest cardinality of such a partition is in $O(n^{\varepsilon/2})$. We postpone this proof and now apply Lemma 3 for the N_i' that are in the range $1, ..., O(n^\varepsilon)$, so that $\sum_i(\max N_i' - \min N_i')^2 = O(n^{1/2+2\varepsilon})$ and

$$\mathbf{P}\left(\sum N_i' - \mathbf{E}\left[\sum N_i'\right] \geqslant c_2\sqrt{n}\right) \leqslant \exp\left(\frac{-c_2^2 n}{O(n^{\varepsilon/2})O(n^{1/2+2\varepsilon})}\right) \leqslant e^{-\Omega(n^{(1-5\varepsilon)/2})}.$$

[7] Intuitively, when we fix the depth h of the least deep point in the witness, the size of the collector is fixed and the points in the collector follow a Poisson process in the part of the collector at depth larger than h, so theorems on Poisson processes can be applied.

Hence, for $\varepsilon < 1/5$, the first term of Eq. (1) is exponentially small. Summing the two terms of Eq. (1) thus gives that the expected size of the maximum silhouette for all viewpoints is in $O(\sqrt{n})$.

It remains to prove that, because $\max d_i \leqslant n^\varepsilon$, the smallest cardinality χ of a partition of the N'_i in subsets of independent variables is in $O(n^{\frac{\varepsilon}{2}})$. An upper bound χ_0 on the number of collectors that can overlap a given collector will also be an upper bound on χ. Indeed, the χ_0 subsets of N'_i defined for $j \in \{1, \ldots, \chi_0\}$ by $\{N'_{j+k\chi_0}, j + k\chi_0 \leqslant \sqrt{n}\}$ are associated to disjoint collectors and hence independent variables N'_i. To compute χ_0, one has to bound the number of ranges R_i contained in a cap of height $h = O(n^\varepsilon/n)$. The length of an arc of height h is $2\arccos(1-h)$ and the length of the arc associated to a range is $2\pi/\sqrt{n}$, hence $\chi_0 \leqslant \frac{2\arccos(1-h)}{2\pi/\sqrt{n}} = \frac{\sqrt{n}}{\pi}\arccos(1-h) \leqslant \frac{1}{\sqrt{2}}\sqrt{nh}$, since for $0 < h < 2$, $\arccos(1-h) < \frac{\pi}{\sqrt{2}}\sqrt{h}$; with $h = O(n^{\varepsilon-1})$, χ_0 and thus χ is in $O(n^{\frac{\varepsilon}{2}})$.

Lower bound. Consider a fixed viewpoint direction Δ_0 and the corresponding witnesses. Each witness has probability $\Theta(1)$ of having height $\frac{1}{n}$; indeed, the area of the associated spherical cap is $\Theta(\frac{1}{n})$, thus the expected number of vertices in it is a constant $\xi \neq 0$, and the probability that the spherical cap contains $k = 0$ vertices is $\frac{\xi^k}{k!}e^{-\xi} = e^{-\xi} < 1$. There are $\Theta(\sqrt{n})$ witnesses, so the expected number of witnesses of height $\frac{1}{n}$ is $\Theta(\sqrt{n})$ (since it is can be seen as the expectation of a binomial). Every such witness contains at least one point on the convex hull and the witnesses are pairwise disjoint, thus the expected size of the silhouette in direction Δ_0 is $\Omega(\sqrt{n})$. It follows that the expectation of the maximum size over all directions is also $\Omega(\sqrt{n})$.

5 Conclusion

We list, in conclusion, some open problems. We proved our bound on the worst-case silhouette for polytopes defined as the convex hull of points sampled according to a Poisson distribution on the sphere. A natural extension would be to consider a Poisson distribution in the ball or points uniformly distributed on the sphere or in the ball. A probably more difficult question is to extend our result to viewpoints that are not necessarily at infinity. One can also consider extensions to higher dimensions. Finally, the question of concentration is also of interest. On this issue, it could be observed from our proof that the probability that the size of the worst-case silhouette is larger than $\delta\sqrt{n}$ is exponentially small in n (with δ defined as in the proof of Section 4). However, this does not give an estimate of the variance of the size of the worst-case silhouette. As a matter of fact, we do not even know an asymptotic equivalent for the expectation.

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