DRAWING PLANAR GRAPHS WITH MANY COLLINEAR VERTICES*

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ABSTRACT. Consider the following problem: Given a planar graph G, what is the maximum number p such that G has a planar straight-line drawing with p collinear vertices? This problem resides at the core of several graph drawing problems, including universal point subsets, untangling, and column planarity. The following results are known for it: Every n-vertex planar graph has a planar straight-line drawing with $\Omega(\sqrt{n})$ collinear vertices; for every n, there is an n-vertex planar graph whose every planar straight-line drawing has $O(n^\sigma)$ collinear vertices, where $\sigma < 0.986$; every n-vertex planar graph of treewidth at most two has a planar straight-line drawing with $\Theta(n)$ collinear vertices.

We extend the linear bound to planar graphs of treewidth at most three and to triconnected cubic planar graphs. This (partially) answers two open problems posed by Ravsky and Verbitsky [*WG* 2011:295–306]. Similar results are not possible for all bounded-treewidth planar graphs or for all bounded-degree planar graphs. For planar graphs of treewidth at most three, our results also imply asymptotically tight bounds for all of the other above mentioned graph drawing problems.

1 Introduction

A set $S \subseteq V(G)$ of vertices in a planar graph G is a *collinear set* if G has a planar straight-line drawing where all the vertices in S are collinear. Ravsky and Verbitsky [21] consider the problem of determining the maximum cardinality of collinear sets in planar graphs. A stronger notion is defined as follows: a set $R \subseteq V(G)$ is a *free collinear set* if a total order $<_R$ of R exists such that, given any set of $|R|$ points on a line ℓ, the graph G has a planar straight-line drawing where the vertices in R are mapped to the given points and their order on ℓ matches the order $<_R$. Free collinear sets were first used (although not named) by Bose *et al.* [4]; also, they were called *free sets* by Ravsky and Verbitsky [21]. Clearly, every free collinear set is also a collinear set. In addition to this obvious relationship to collinear sets, free collinear sets have connections to other graph drawing problems, as will be discussed later in this introduction.

Based on the results in [4], Dujmović [9] showed that every n-vertex planar graph has a free collinear set of cardinality at least $\sqrt{n/2}$. A natural question to consider would be whether a linear bound is possible for all planar graphs. Ravsky and Verbitsky [21] provided a negative answer to that question. In particular, they observed that if a planar triangulation has a large collinear set, then its dual graph has a cycle of proportional length. Since there are m-vertex triconnected cubic planar graphs whose longest cycle has length $O(m^\sigma)$ [13], it follows that there are n-vertex planar graphs in which the cardinality of every collinear set is $O(n^\sigma)$. Here σ is a known graph-theoretic constant called *shortness exponent*, for which the best known upper bound is $\sigma < 0.986$ [13].

In addition to the natural open problem of closing the gap between the $\Omega(n^{0.5})$ and $O(n^\sigma)$ bounds for general n-vertex planar graphs, these results raise the question of which classes of planar graphs have (free) collinear sets of linear cardinality. Goaoc *et al.* [11] proved (implicitly) that n-vertex outerplanar graphs have free collinear sets of cardinality $(n+1)/2$; this result was explicitly stated and proved by Ravsky and Verbitsky [21]. Ravsky and Verbitsky [21] also considerably strengthened that result by proving that all n-vertex planar graphs of treewidth at most two have free collinear sets of cardinality $n/30$; they also asked for other classes of graphs with (free) collinear sets of linear cardinality, calling special attention to planar graphs of bounded treewidth and to planar graphs of bounded degree. In this paper we prove the following results:

1. every n-vertex planar graph of treewidth at most three has a free collinear set with cardinality $\lceil \frac{n-3}{8} \rceil$;

2. every n-vertex triconnected cubic planar graph has a collinear set with cardinality $\lceil \frac{n}{4} \rceil$; and

3. every planar graph of treewidth k has a collinear set with cardinality $\Omega(k^2)$.

Our first result generalizes the previous result on planar graphs of treewidth at most two [21]. As noted by Ravsky and Verbitsky in the full version of their paper [22, Corollary 3.5], there are n-vertex planar graphs of treewidth at most 8 whose largest collinear set has cardinality $o(n)$. In order to obtain that, the authors show a construction relying on the dual of the Barnette-Bosák-Lederberg's non-Hamiltonian cubic triconnected planar graph. We observe that the Tutte's graph, which is also non-Hamiltonian, cubic, planar, and triconnected, has a dual graph that has treewidth 5. Hence, by exploiting the same construction proposed by Ravsky and Verbitsky, however by relying on the dual of the Tutte's graph rather than on the dual of the Barnette-Bosák-Lederberg's graph, we get that the sub-linear upper bound for the size of the largest collinear set holds true for planar graphs of treewidth at most 5. Thus, our first result leaves $k = 4$ as the only remaining open case for the question of whether planar graphs of treewidth at most k admit (free) collinear sets with linear cardinality.

Our second result provides the first linear lower bound on the cardinality of collinear sets for a fairly wide class of bounded-degree planar graphs. The result cannot be extended to all bounded-degree planar graphs. In particular it cannot be extended to planar graphs of degree at most 7, since there exist n-vertex planar triangulations of maximum degree 7 whose dual graph has a longest cycle of length $o(n)$ [18].

Finally, our third result improves the $\Omega(\sqrt{n})$ bound on the cardinality of collinear sets in general planar graphs for all planar graphs whose treewidth is $\omega(\sqrt[4]{n})$.

We now discuss applications of our results to other graph drawing problems. Since our first result gives *free* collinear sets, its consequences are broader.

A *column planar set* in a planar graph G is a set $Q \subseteq V(G)$ satisfying the following property: there exists a function $\gamma : Q \to \mathbb{R}$ such that, for any function $\lambda : Q \to \mathbb{R}$, there exists a planar straight-line drawing of G in which each vertex $v \in Q$ is mapped to point $(\gamma(v), \lambda(v))$. Column planar sets were defined by Barba *et al.* [2] motivated by applications to partial simultaneous geometric embeddings.[1] They proved that n-vertex trees have column planar sets of size $14n/17$. The lower bounds in all our three results carry over to the size of column planar sets for the corresponding graph classes.

A *universal point subset* for the n-vertex planar graphs is a set P of $k \leq n$ points in the plane such that, for every n-vertex planar graph G, there exists a planar straight-line drawing of G in which k vertices are placed at the k points in P. Universal point subsets were introduced by Angelini *et al.* [1]. Every set of n points in general position is a universal point subset for the n-vertex outerplanar graphs [12, 3, 6] and every set of $\sqrt{n/2}$ points in the plane is a universal point subset for the n-vertex planar graphs [9]. As a corollary of our first result, we obtain that every set of $\lceil \frac{n-3}{8} \rceil$ points in the plane is a universal point subset for the n-vertex planar graphs of treewidth at most three.

Given a straight-line drawing of a planar graph, possibly with crossings, to *untangle* it means to assign new locations to some of its vertices so that the resulting straight-line drawing is planar. The goal is to do so while *keeping fixed* (i.e., not changing the location of) as many vertices as possible. Several papers have studied the untangling problem [19, 5, 8, 4, 11, 16, 21]. It is known that general n-vertex planar graphs can be untangled while keeping $\Omega(n^{0.25})$ vertices fixed [4] and that there are n-vertex planar graphs that cannot be untangled while keeping $\Omega(n^{0.4948})$ vertices fixed [5]. Asymptotically tight bounds are known for paths [8], trees [11], outerplanar graphs [11], and planar graphs of treewidth two [21]. As a corollary of our first result, we obtain that every n-vertex planar graph of treewidth at most three can be untangled while keeping $\Omega(\sqrt{n})$ vertices fixed. This bound is the best possible, as there are forests of stars that cannot be untangled while keeping $\omega(\sqrt{n})$ vertices fixed [4]. Our result generalizes previous results on trees, outerplanar graphs and planar graphs of treewidth at most two.

2   Preliminaries

The graphs called *k-trees* are defined recursively as follows. A complete graph on $k + 1$ vertices is a k-tree. If G is a k-tree, then the graph obtained by adding a new vertex to G and making it adjacent to all the vertices in a k-clique of G is a k-tree. The *treewidth* of a graph G is the minimum k such that G is a subgraph of some k-tree.

A connected *plane graph* G is a connected planar graph together with a *plane em-*

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[1] The original definition by Barba *et al.* [2] had also an extra condition that required the point set composed of the points $(\gamma(v), \lambda(v))$ for all $v \in Q$ not to have three points on a line.

bedding, that is, an equivalence class of planar drawings of G, where two planar drawings are *equivalent* if they have the same *rotation system* (i.e., the same clockwise order of the edges incident to each vertex) and the same *outer face* (i.e., the unbounded face is delimited by the same walk). Note that two equivalent planar drawings can be obtained one from the other via a homeomorphism of the plane to itself. We always think about a plane graph G as if it is drawn according to its plane embedding; also, when we talk about a planar drawing of G, we always mean that it respects the plane embedding of G. The *interior* of G is the closure of the union of the internal faces of G. We associate with a subgraph H of G the plane embedding obtained from the one of G by deleting vertices and edges not in H.

The degree of a vertex v in a graph G is denoted by $\delta_G(v)$. A graph is *cubic* (*subcubic*) if every vertex has degree 3 (resp. at most 3). Let G be a graph and $U \subseteq V(G)$. Let $G - U$ be the graph obtained from G by removing the vertices in U and their incident edges. The subgraph of G *induced* by U has U as vertex set and has an edge $e \in E(G)$ if and only if both its end-vertices are in U. Let H be a subgraph of G; then H is *induced* if H is induced by $V(H)$. If $v \in V(G) - V(H)$, then let $H \cup \{v\}$ be the subgraph of G composed of H and of the isolated vertex v. An *H-bridge* B is either a *trivial H-bridge* – an edge of G not in H with both end-vertices in H – or a non-trivial H-bridge – a connected component of $G - V(H)$ together with the edges from that component to H; the vertices in $V(H) \cap V(B)$ are called *attachments*.

Let G be a connected graph. A *cut-vertex* is a vertex whose removal disconnects G. If G has no cut-vertex and it is not a single edge, then it is *biconnected*. A *biconnected component* of G is a maximal (with respect to both vertices and edges) biconnected subgraph of G. Let G be a biconnected graph. A *separation pair* is a pair of vertices whose removal disconnects G. If G has no separation pair, then it is *triconnected*. Given a separation pair $\{a, b\}$ in a biconnected graph G, an *$\{a, b\}$-component* is either a *trivial $\{a, b\}$-component* – the edge (a, b) – or a *non-trivial $\{a, b\}$-component* – a subgraph of G induced by a, b, and the vertices of a connected component of $G - \{a, b\}$.

3   From a Geometric to a Topological Problem

In this section we show that the problem of determining a large collinear set in a planar graph, which is geometric by definition, can be transformed into a purely topological problem. This result may be useful to obtain bounds for the size of collinear sets in classes of planar graphs different from the ones we studied in this paper.

A curve is *simple* if it does not self-intersect. Given a planar drawing Γ of a plane graph G, we say that an open simple curve λ is *good* for Γ if, for each edge e of G, the curve λ either entirely contains e or has at most one point in common with e (if λ passes through an end-vertex of e, that counts as a common point). Clearly, the existence of a good curve passing through a certain sequence of vertices, edges, and faces of G does not depend on the actual drawing Γ, but only on the plane embedding of G. For this reason we often talk about the existence of good curves in plane graphs, rather than in their planar drawings. Let $R_{G,\lambda}$ be the only unbounded region of the plane defined by G and λ. The curve λ is *proper* if both its end-points are incident to $R_{G,\lambda}$. We have the following.

Theorem 1. *A plane graph G has a planar straight-line drawing with χ collinear vertices if and only if G has a proper good curve that passes through χ vertices of G.*

Proof. For the necessity, assume that G has a planar straight-line drawing Γ with χ vertices lying on a common line ℓ. We transform ℓ into a straight-line segment λ by cutting off two disjoint half-lines of ℓ in the outer face of G. This immediately implies that λ is proper. Further, λ passes through χ vertices of G since ℓ does. Finally, if an edge e has two common points with λ, then λ entirely contains it, since λ and e are straight-line segments in Γ.

For the sufficiency, assume that G has a proper good curve λ passing through χ of its vertices; see Fig. 1(a). Augment G by adding to it (refer to Fig. 1(b)): (i) a dummy vertex at each proper crossing between an edge and λ; (ii) two dummy vertices at the end-points a and b of λ; (iii) an edge between any two consecutive vertices of G along λ, which now represents a path $(a, \ldots, b)$ of G; (iv) two dummy vertices d_1 and d_2 in $R_{G,\lambda}$; and (v) edges in $R_{G,\lambda}$ connecting each of d_1 and d_2 with each of a and b so that the cycles $C_1 = (d_1, a, \ldots, b)$ and $C_2 = (d_2, a, \ldots, b)$ are embedded in this counter-clockwise and clockwise direction in G, respectively. For $i = 1, 2$, let G_i be the subgraph of G induced by the vertices of C_i or inside it. Triangulate the internal faces of G_i with dummy vertices and edges, so that there are no edges between non-consecutive vertices of C_i. This is possible because these edges do not exist in the original graph G, given that λ is good.

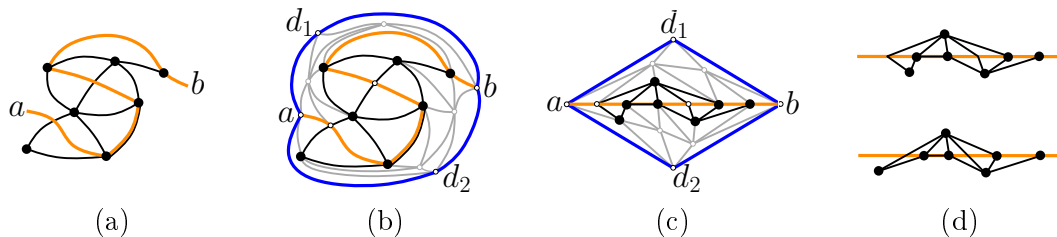|			
(a)	(b)	(c)	(d)

Figure 1: (a) A proper good curve λ (orange) for a plane graph G (black). (b) Augmentation of G with dummy vertices and edges. (c) A planar straight-line drawing of the augmented graph G. (d) Planar polyline (top) and straight-line (bottom) drawings of the original graph G.

Represent C_1 as a convex polygon Q_1 whose all vertices, except for d_1, lie along a horizontal line ℓ, with a to the left of b and d_1 above ℓ; see Fig. 1(c). The graph G_1 is triconnected, as it contains no edge between any two non-consecutive vertices of its only non-triangular face.[2] Thus, there exists a planar straight-line drawing of G_1 in which C_1

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[2]This implication can be proved as follows. Suppose that G_1 is not triconnected; we prove that there exists an edge between two non-consecutive vertices along C_1. Augment G_1 to a graph G_1^+ by adding, in the outer face of G_1, a vertex v^* and edges between v^* and all the vertices of C_1. Since the outer face of G_1 was its only non-triangular face, we have that G_1^+ is a maximal planar graph. It follows that G_1^+ is 3-connected and every 3-cut $\{u, v, z\}$ of G_1^+ induces a cycle (u, v, z) [14]. Now, for any vertex 2-cut $\{s, t\}$ of G_1, we have that $\{s, t, v^*\}$ is a vertex 3-cut of G_1^+, hence (s, t, v^*) is a cycle of G_1^+; it follows that the edge (s, t) belongs to G_1 and that s and t are vertices of C_1. Finally, the fact that s and t are not consecutive along C_1 follows from the fact that the cycle (s, t, v^*) contains vertices of G_1^+ on both sides, given that $\{s, t, v^*\}$ is a vertex 3-cut of G_1^+.

is represented by Q_1 [24]. Analogously, represent C_2 as a convex polygon Q_2 whose all vertices, except for d_2, lie at the same points as in Q_1, with d_2 below ℓ. Construct a planar straight-line drawing of G_2 in which C_2 is represented by Q_2.

Removing the dummy vertices and edges results in a planar drawing Γ of the original graph G in which each edge e is a y-monotone curve; see Fig. 1(d). In particular, the fact that λ crosses e at most once ensures that e is either a straight-line segment or is composed of two straight-line segments that are one below and one above ℓ and that share an end-point on ℓ. As proved in [10, 20], there exists a planar straight-line drawing Γ' of G in which the y-coordinate of each vertex is the same as in Γ. Since λ passes through χ vertices of G, we have that χ vertices of G lie along ℓ in Γ'. $\qquad\square$

Theorem 1 can be stated for planar graphs without a given plane embedding as follows: A planar graph has a collinear set with χ vertices if and only if it admits a plane embedding for which a proper good curve can be drawn that passes through χ of its vertices. While this version of Theorem 1 might be more general, it is less useful for us, so we preferred explicitly stating its version for plane graphs.

4 Planar Graphs with Treewidth at most Three

In this section we prove the following theorem.

Theorem 2. *Every n-vertex plane graph of treewidth at most three admits a planar straight-line drawing with at least $\lceil \frac{n-3}{8} \rceil$ collinear vertices.*

For technical reasons, we regard a plane cycle with three vertices as a plane 3-tree. Then every plane graph G with $n \geq 3$ vertices and treewidth at most three can be augmented with dummy edges to a plane 3-tree G' [17] which is a plane triangulation. A planar straight-line drawing of G with $\lceil \frac{n-3}{8} \rceil$ collinear vertices can be obtained from a planar straight-line drawing of G' with $\lceil \frac{n-3}{8} \rceil$ collinear vertices by removing the inserted dummy edges. Thus for the remainder of this section, we assume that G is a plane 3-tree.

By Theorem 1 it suffices to prove that G admits a proper good curve passing through at least $\lceil \frac{n-3}{8} \rceil$ vertices of G. Let u, v, and z be the external vertices of G. If $n = 3$, then G does not contain any internal vertex and we say that it is *hollow*. If G is not hollow, let w be the unique internal vertex of G adjacent to all of u, v, and z; we say that w is the *central vertex* of G. Let G_1, G_2, and G_3 be the plane 3-trees which are the subgraphs of G whose outer faces are delimited by the cycles (u, v, w), (u, z, w), and (v, z, w). We call G_1, G_2, and G_3 *children* of G and *children* of w.

We associate to each internal vertex x of G a plane 3-tree $G(x)$, which is a subgraph of G, as follows. We associate G to w and we recursively associate plane 3-trees to the internal vertices of the children G_1, G_2, and G_3 of G. Note that x is the central vertex of the plane 3-tree $G(x)$ associated to it.

We now introduce a classification of the internal vertices of G; see Fig. 2(a). Consider an internal vertex x of G. We say that x is of *type A*, *B*, *C*, or *D* if, respectively, 3, 2, 1,

or 0 of the children of $G(x)$ are hollow. Let $a(G)$, $b(G)$, $c(G)$, and $d(G)$ be the number of internal vertices of G of type A, B, C, and D, respectively. Let $m = n - 3$ be the number of internal vertices of G.
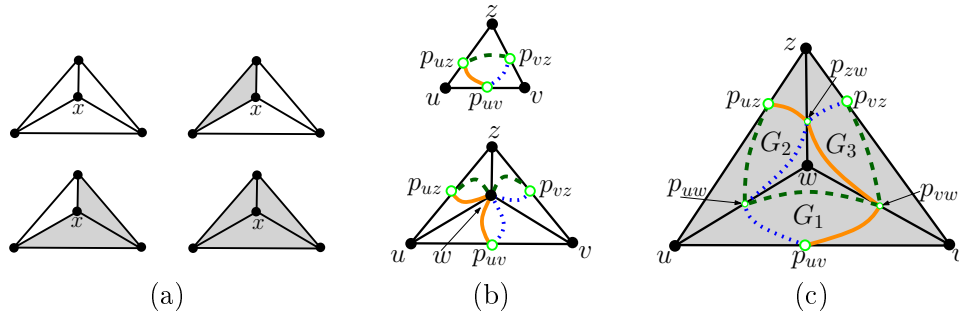

Figure 2: (a) A vertex x of type A (top-left), B (top-right), C (bottom-left), and D (bottom-right). (b) The curves $\lambda_u(G)$ (solid), $\lambda_v(G)$ (dotted), and $\lambda_z(G)$ (dashed) if $m = 0$ (top) and $m = 1$ (bottom). (c) The curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ if w is of type C or D.

In the following we present an algorithm that constructs three proper good curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ lying in the interior of G. For every edge (x, y) of G, let p_{xy} be an arbitrary internal point of (x, y). The end-points of $\lambda_u(G)$ are p_{uv} and p_{uz}, the end-points of $\lambda_v(G)$ are p_{uv} and p_{vz}, and the end-points of $\lambda_z(G)$ are p_{uz} and p_{vz}. Although each of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ is a good curve, any two of these curves might cross each other arbitrarily and might pass through the same vertices of G. Each of these curves passes through all the internal vertices of G of type A, through no vertex of type C or D, and through "some" vertices of type B. We will prove that the total number of internal vertices of G through which the curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ pass is at least $\frac{3m}{8}$, hence one of them passes through at least $\lceil \frac{m}{8} \rceil$ internal vertices of G.

The curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ are constructed by induction on m. In the base case we have $m \leq 1$; refer to Fig. 2(b). If $m = 0$, then $\lambda_u(G)$ starts at p_{uv}, traverses the internal face (u, v, z) of G, and ends at p_{uz}. The curves $\lambda_v(G)$ and $\lambda_z(G)$ are constructed analogously. If $m = 1$, then $\lambda_u(G)$ starts at p_{uv}, traverses the internal face (u, v, w) of G, passes through the central vertex w of G, traverses the internal face (u, z, w) of G, and ends at p_{uz}. The curves $\lambda_v(G)$ and $\lambda_z(G)$ are constructed analogously.

In the inductive case we have $m > 1$ and the central vertex w of G is of one of types B–D. We outline our strategy for constructing $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$. If w is of type C or D, then proper good curves are inductively constructed for the children of G and composed to obtain $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$. If w is of type B, then a sequence of vertices of type B is considered; this sequence is called a *B-chain*. The B-chain starts at $w_1 = w$; while the only child H_i of the last vertex w_i in the sequence has a central vertex w_{i+1} of type B, the sequence is enriched with w_{i+1}; once w_{i+1} is not of type B, induction is applied on H_i, and the three curves obtained by induction are composed with curves passing through vertices of the B-chain to get $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$.

We now detail our construction. Assume first that w is of type C or D. Refer to

external vertices of H_1	P_u	P_v	P_z
v, w, z	(u, w)	(v)	(z)
u, w, z	(u)	(v, w)	(z)
u, v, w	(u)	(v)	(z, w)

Table 1: Definition of P_u, P_v, and P_z depending on the external vertices of H_1.

Fig. 2(c). Inductively construct curves $\lambda_u(G_1)$, $\lambda_v(G_1)$, and $\lambda_w(G_1)$ for G_1, curves $\lambda_u(G_2)$, $\lambda_z(G_2)$, and $\lambda_w(G_2)$ for G_2, and curves $\lambda_v(G_3)$, $\lambda_z(G_3)$, and $\lambda_w(G_3)$ for G_3. Let

$$
\begin{aligned}
\lambda_u(G) &= \lambda_v(G_1) \cup \lambda_w(G_3) \cup \lambda_z(G_2), \\
\lambda_v(G) &= \lambda_u(G_1) \cup \lambda_w(G_2) \cup \lambda_z(G_3), \text{ and} \\
\lambda_z(G) &= \lambda_u(G_2) \cup \lambda_w(G_1) \cup \lambda_v(G_3).
\end{aligned}
$$

Next, consider the case in which w is of type B. In order to describe how to construct curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$, we need to further explore the structure of G.

Let $H_0 = G$, let $w_1 = w$, and let H_1 be the only non-hollow child of G. We define three paths P_u, P_v, and P_z as described in Table 1, depending on which among u, v, z, and w are the external vertices of H_1.

Now suppose that, for some $i \geq 1$, a sequence $w_1, \ldots, w_i$ of vertices of type B, a sequence $H_0, H_1, \ldots, H_i$ of plane 3-trees, and three paths P_u, P_v, and P_z (possibly single vertices or edges) have been defined so that the following properties hold true. Let u', v', and z' be the three external vertices of H_i. Let K_i be the subgraph of G induced by u, v, z, u', v', z', and by the vertices that lie inside the cycle (u, v, z) and outside the cycle (u', v', z').

(1) For $1 \leq j \leq i$, the vertex w_j is the central vertex of H_{j-1} and H_j is the only non-hollow child of H_{j-1};

(2) P_u, P_v, and P_z are vertex-disjoint and each of them is induced in G;

(3) P_u, P_v, and P_z connect u, v, and z with u', v', and z', respectively;

(4) every vertex of K_i belongs to P_u, P_v, or P_z; and

(5) every edge of K_i either belongs to P_u, P_v, or P_z, or connects two vertices on distinct paths among P_u, P_v, and P_z.

Properties (1)–(5) are indeed satisfied with $i = 1$. Consider the central vertex of H_i and denote it by w_{i+1}.

If w_{i+1} is of type B, then let H_{i+1} be the only non-hollow child of H_i. If the cycle (v', z', w_{i+1}) delimits the outer face of H_{i+1}, then add the edge (u', w_{i+1}) to P_u and leave P_v and P_z unaltered. The cases in which the cycles (u', z', w_{i+1}) or (u', v', w_{i+1}) delimit the outer face of H_{i+1} can be dealt with analogously. Properties (1)–(5) are clearly satisfied by the described construction. In particular, every vertex of K_{i+1} belongs to P_u, P_v, or P_z, given that (i) every vertex of K_i belongs to P_u, P_v, or P_z, (ii) w_{i+1} belongs to P_u, and (iii) the children of H_i different from H_{i+1} are hollow. Further, every edge of K_{i+1} either belongs to P_u, P_v, or P_z, or connects two vertices on distinct paths among P_u, P_v, and P_z, given

that (i) every edge of K_i either belongs to P_u, P_v, or P_z, or connects two vertices on distinct paths among P_u, P_v, and P_z, (ii) the edge (u', w_{i+1}) belongs to P_u, (iii) the edge (v', w_{i+1}) connects a vertex of P_v with a vertex of P_u, (iv) the edge (z', w_{i+1}) connects a vertex of P_z with a vertex of P_u, and (v) the children of H_i different from H_{i+1} are hollow.

If w_{i+1} is not of type B, then we call the sequence $w_1, \ldots, w_i$ a *B-chain* of G; note that all of $w_1, \ldots, w_i$ are of type B. For the sake of the simplicity of notation, let $H = H_i$. We label the vertices of P_u, P_v, and P_z as follows.

- $P_u = (u = u_1, u_2, \ldots, u_U = u')$, where U is the number of vertices of P_u;

- $P_v = (v = v_1, v_2, \ldots, v_V = v')$, where V is the number of vertices of P_v; and

- $P_z = (z = z_1, z_2, \ldots, z_Z = z')$, where Z is the number of vertices of P_z.

We also define the following cycles.

- $C_{uv} = P_u \cup (u, v) \cup P_v \cup (u', v')$;

- $C_{uz} = P_u \cup (u, z) \cup P_z \cup (u', z')$; and

- $C_{vz} = P_v \cup (v, z) \cup P_z \cup (v', z')$.

None of the cycles C_{uv}, C_{uz}, and C_{vz} contains a vertex in its interior; further, every edge in the interior of C_{uv}, C_{uz}, or C_{vz} connects two vertices on distinct paths among P_u, P_v, and P_z. Indeed, every vertex or edge in the interior of C_{uv}, C_{uz}, or C_{vz} is an internal vertex or an internal edge of K_i, respectively; then the two statements follow from Properties (4) and (5).

We are going to use the following (a similar lemma can be stated for C_{uz} and C_{vz}).

Lemma 1. *Let p_1 and p_2 be two points on the boundary of C_{uv}, possibly coinciding with vertices of C_{uv}, and not both on the same edge of G. There exists a good curve connecting p_1 and p_2, lying inside C_{uv}, except at its end-points, and intersecting every edge of G inside C_{uv} at most once.*

Proof. The lemma has a simple geometric proof. Represent C_{uv} as a strictly-convex polygon and draw the edges of G inside C_{uv} as straight-line segments. Then the straight-line segment $\overline{p_1 p_2}$ is a good curve satisfying the requirements of the lemma. $\square$

We now describe how to construct curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$. First, inductively construct curves $\lambda_{u'}(H)$, $\lambda_{v'}(H)$, and $\lambda_{z'}(H)$ for H. The construction of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ varies based on how many among P_u, P_v, and P_z are single vertices. Observe that not all of P_u, P_v, and P_z are single vertices, as $w_1 \notin \{u, v, z\}$.

Suppose first that none of P_u, P_v, and P_z is a single vertex, as in Fig. 3(a). We describe how to construct $\lambda_u(G)$, as the construction of $\lambda_v(G)$ and $\lambda_z(G)$ is analogous.
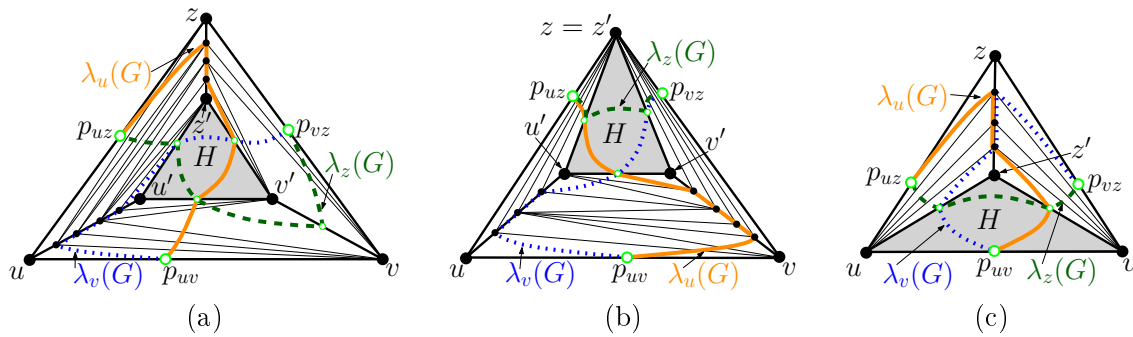
Figure 3: Construction of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ if w is of type B. (a) None of P_u, P_v, and P_z is a single vertex. (b) P_z is a single vertex while P_u and P_v are not. (c) P_u and P_v are single vertices while P_z is not.

- If $Z > 2$, then $\lambda_u(G)$ consists of curves $\lambda_u^0, \ldots, \lambda_u^4$. The curve λ_u^0 lies inside C_{uz} and connects p_{uz} with z_2, which is internal to P_z since $Z > 2$; the curve λ_u^1 coincides with the path $(z_2, \ldots, z_{Z-1})$ (the path consists of a single vertex if $Z = 3$); the curve λ_u^2 lies inside C_{vz} and connects z_{Z-1} with $p_{v'z'}$; the curve λ_u^3 coincides with $\lambda_{v'}(H)$; finally, λ_u^4 lies inside C_{uv} and connects $p_{u'v'}$ with p_{uv}. The curves λ_u^0, λ_u^2, and λ_u^4 are constructed as in Lemma 1.

- If $Z = 2$, then λ_u consists of curves $\lambda_u^1, \ldots, \lambda_u^4$. The curve λ_u^1 lies inside C_{uz} and connects p_{uz} with $p_{zz'}$; the curve λ_u^2 lies inside C_{vz} and connects $p_{zz'}$ with $p_{v'z'}$; the curves λ_u^3 and λ_u^4 are defined as in the case $Z > 2$. The curves λ_u^1, λ_u^2, and λ_u^4 are constructed as in Lemma 1.

Suppose next that one of P_u, P_v, and P_z, say P_z, is a single vertex, as in Fig. 3(b). We describe how to construct $\lambda_u(G)$ and $\lambda_z(G)$; the construction of $\lambda_v(G)$ is analogous to the one of $\lambda_u(G)$. The curve $\lambda_z(G)$ consists of curves $\lambda_z^0, \lambda_z^1, \lambda_z^2$. The curve λ_z^0 lies inside C_{uz} and connects p_{uz} with $p_{u'z}$; the curve λ_z^1 coincides with $\lambda_{z'}(H)$; the curve λ_z^2 lies inside C_{vz} and connects $p_{v'z}$ with p_{vz}. The curves λ_z^0 and λ_z^2 are constructed as in Lemma 1. The curve $\lambda_u(G)$ is constructed as follows.

- If $V > 2$, then $\lambda_u(G)$ consists of curves $\lambda_u^0, \ldots, \lambda_u^4$. The curve λ_u^0 lies inside C_{uv} and connects p_{uv} with v_2, which is internal to P_v since $V > 2$; the curve λ_u^1 coincides with the path $(v_2, \ldots, v_{V-1})$ (the path consists of a single vertex if $V = 3$); the curve λ_u^2 lies inside C_{uv} and connects v_{V-1} with $p_{u'v'}$; the curve λ_u^3 coincides with $\lambda_{u'}(H)$; finally, λ_u^4 coincides with λ_z^0. The curves λ_u^0, λ_u^2, and λ_u^4 are constructed as in Lemma 1.

- If $V = 2$, then $\lambda_u(G)$ consists of curves $\lambda_u^0, \lambda_u^1, \lambda_u^2$. The curve λ_u^0 lies inside C_{uv} and connects p_{uv} with $p_{u'v'}$; the curve λ_u^1 coincides with $\lambda_{u'}(H)$; the curve λ_u^2 coincides with λ_z^0. The curves λ_u^0 and λ_u^2 are constructed as in Lemma 1.

Suppose finally that two of P_u, P_v, and P_z, say P_u and P_v, are single vertices, as in Fig. 3(c). We describe how to construct $\lambda_u(G)$ and $\lambda_z(G)$; the construction of $\lambda_v(G)$ is

analogous to the one of $\lambda_u(G)$. The curve $\lambda_z(G)$ consists of curves $\lambda_z^0, \lambda_z^1, \lambda_z^2$. The curve λ_z^0 lies inside C_{uz} and connects p_{uz} with $p_{uz'}$; the curve λ_z^1 coincides with $\lambda_{z'}(H)$; the curve λ_z^2 lies inside C_{vz} and connects $p_{vz'}$ with p_{vz}. The curves λ_z^0 and λ_z^2 are constructed as in Lemma 1. The curve $\lambda_u(G)$ is constructed as follows.

- If $Z > 2$, then $\lambda_u(G)$ consists of curves $\lambda_u^0, \ldots, \lambda_u^3$. The curve λ_u^0 lies inside C_{uz} and connects p_{uz} with z_2, which is internal to P_z since $Z > 2$; the curve λ_u^1 coincides with the path $(z_2, \ldots, z_{Z-1})$ (the path consists of a single vertex if $Z = 3$); the curve λ_u^2 lies inside C_{vz} and connects z_{Z-1} with $p_{vz'}$; finally, the curve λ_u^3 coincides with $\lambda_{v'}(H)$. The curves λ_u^0 and λ_u^2 are constructed as in Lemma 1.

- If $Z = 2$, then $\lambda_u(G)$ consists of curves $\lambda_u^0, \lambda_u^1, \lambda_u^2$. The curve λ_u^0 lies inside C_{uz} and connects p_{uz} with $p_{zz'}$; the curve λ_u^1 lies inside C_{vz} and connects $p_{zz'}$ with $p_{vz'}$; the curve λ_u^2 coincides with $\lambda_{v'}(H)$. The curves λ_u^0 and λ_u^1 are constructed as in Lemma 1.

This completes the construction of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$. Since these curves lie in the interior of G and since their end-points are incident to the outer face of G, they are proper. We now prove that they are good and pass through many vertices of G.

Lemma 2. *The curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ are good.*

Proof. We prove that $\lambda_u(G)$ is good by induction on m; the proof for $\lambda_v(G)$ and $\lambda_z(G)$ is analogous. If $m \leq 1$, then the statement is trivial. If $m > 1$, then the central vertex w of G is of one of types B–D.

If w is of type C or D, then $\lambda_u(G)$ is composed of the three curves $\lambda_v(G_1)$, $\lambda_w(G_3)$, and $\lambda_z(G_2)$, each of which is good by induction. By construction, $\lambda_u(G)$ intersects the edges (u, v), (v, w), (z, w), and (u, z) at the points p_{uv}, p_{vw}, p_{zw}, and p_{uz}, respectively, and does not intersect the edges (v, z) and (u, w) at all. Consider any edge e internal to G_1. The curves $\lambda_w(G_3)$ and $\lambda_z(G_2)$ have no intersection with the region of the plane inside the cycle (u, v, w); further, $\lambda_u(G)$ does not pass through u, v, or w. Hence, $\lambda_u(G)$ contains e or intersects e at most once, given that $\lambda_v(G_1)$ is good. Analogously, for every internal edge e of G_2 and G_3, we have that $\lambda_u(G)$ contains e or intersects e at most once.

Assume now that w is of type B. We prove that, for every edge e of G, the curve $\lambda_u(G)$ either contains e or intersects e at most once.

- By construction, $\lambda_u(G)$ intersects each of (u, v), (u, z), (v, z), (u', v'), (u', z'), and (v', z') at most once. Also, $\lambda_u(G)$ has no intersection with any edge of the path P_u.

- Consider an edge e internal to H. The curves that compose $\lambda_u(G)$ and that lie inside C_{uv}, C_{uz}, or C_{vz}, or that coincide with a subpath of P_v or P_z have no intersection with the region of the plane inside the cycle (u', v', z'); further, $\lambda_u(G)$ does not pass through u', v', or z'. Hence, $\lambda_u(G)$ contains e or intersects e at most once, given that $\lambda_{u'}(H)$, $\lambda_{v'}(H)$, and $\lambda_{z'}(H)$ are good.

- Consider an edge $e = (v_j, v_{j+1}) \in P_v$ (the argument for the edges in P_z is analogous). If $\lambda_u(G)$ has no intersection with P_v, then it has no intersection with e. If $\lambda_u(G)$

intersects P_v and $V > 2$, then it contains e (if $2 \leq j \leq V-2$), or it intersects e only at v_{j+1} (if $j = 1$), or it intersects e only at v_j (if $j = V - 1$). Finally, if $\lambda_u(G)$ intersects P_v and $V = 2$, then $\lambda_u(G)$ properly crosses e at $p_{vv'}$.

- We prove that $\lambda_u(G)$ intersects the edges inside C_{uv} at most once (the argument for the edges inside C_{uz} or C_{vz} is analogous). Recall that, since P_u and P_v are induced, every edge inside C_{uv} connects a vertex of P_u and a vertex of P_v. Assume that $\lambda_u(G)$ contains a curve λ_u^0 inside C_{uv} that connects p_{uv} with v_2, a curve λ_u^1 that coincides with the path $(v_2, \ldots, v_{V-1})$, and a curve λ_u^2 inside C_{uv} that connects v_{V-1} with $p_{u'v'}$, as in Fig. 3(b); all the other cases are simpler to handle.

  - Consider any edge e incident to v_1 inside C_{uv}. The curve λ_u^0 intersects e once – in fact the end-points of λ_u^0 alternate with those of e along C_{uv}, hence λ_u^0 intersects e; moreover, λ_u^0 and e do not intersect more than once by Lemma 1. The path $(v_2, \ldots, v_{V-1})$, and hence the curve λ_u^1 that coincides with it, has no intersection with e, since the end-vertices of e are not in $v_2, \ldots, v_{V-1}$. Further, the curve λ_u^2 has no intersection with e – in fact the end-points of λ_u^2 do not alternate with those of e along C_{uv}, hence if λ_u^2 and e intersected, then they would intersect at least twice, which is not possible by Lemma 1. Thus, $\lambda_u(G)$ intersects e once.

  - Analogously, every edge e incident to v_V inside C_{uv} has no intersection with λ_u^0, no intersection with λ_u^1, and one intersection with λ_u^2, hence $\lambda_u(G)$ intersects e once.

  - Finally, consider any edge e incident to v_j, with $2 \leq j \leq V - 1$. The curves λ_u^0 and λ_u^2 have no intersection with e – in fact the end-points of each of these curves do not alternate with those of e along C_{uv}, hence each of these curves does not intersect e by Lemma 1. Further, λ_u^1 contains an end-vertex of e and thus it intersects e once. It follows that $\lambda_u(G)$ intersects e once.

This concludes the proof of the lemma. □

We introduce three parameters. Let $s(G)$ be the total number of vertices of G through which the curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ pass, counting each vertex with a multiplicity equal to the number of curves that pass through it. Further, let $x(G)$ be the number of internal vertices of type B none of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ passes through. Finally, let $h(G)$ be the number of B-chains of G. We have the following inequalities.

Lemma 3. *The following hold true if $m \geq 1$:*

(1) $a(G) + b(G) + c(G) + d(G) = m$;

(2) $a(G) = c(G) + 2d(G) + 1$;

(3) $h(G) \leq 2c(G) + 3d(G) + 1$;

(4) $x(G) \leq b(G)$;

(5) $x(G) \leq 3h(G)$; and

(6) $s(G) \geq 3a(G) + b(G) - x(G)$.

Proof. **(1)** $a(G) + b(G) + c(G) + d(G) = m$. This equality follows from the fact that every internal vertex of G is of one of types A–D.

(2) $a(G) = c(G) + 2d(G) + 1$. We use induction on m. If $m = 1$, then the statement is easily proved, as then the only internal vertex w of G is of type A, hence $a(G) = 1$ and $c(G) = d(G) = 0$. If $m > 1$, then the central vertex w of G is of one of types B–D.

Suppose first that w is of type B. Also, suppose that G_1 has internal vertices; the other cases are analogous. Since w is of type B, we have $a(G) = a(G_1)$, $c(G) = c(G_1)$, and $d(G) = d(G_1)$. Hence, $a(G) = a(G_1) = c(G_1) + 2d(G_1) + 1 = c(G) + 2d(G) + 1$; the second equality holds by induction.

Suppose next that w is of type C. Also, suppose that G_1 and G_2 have internal vertices; the other cases are analogous. Since w is of type C, we have $a(G) = a(G_1) + a(G_2)$, $c(G) = c(G_1) + c(G_2) + 1$, and $d(G) = d(G_1) + d(G_2)$. Hence, $a(G) = a(G_1) + a(G_2) = (c(G_1) + 2d(G_1) + 1) + (c(G_2) + 2d(G_2) + 1) = (c(G_1) + c(G_2) + 1) + 2(d(G_1) + d(G_2)) + 1 = c(G) + 2d(G) + 1$; the second equality holds by induction.

Suppose finally that w is of type D. Then we have $a(G) = a(G_1) + a(G_2) + a(G_3)$, $c(G) = c(G_1) + c(G_2) + c(G_3)$, and $d(G) = d(G_1) + d(G_2) + d(G_3) + 1$. Hence, $a(G) = a(G_1) + a(G_2) + a(G_3) = (c(G_1) + 2d(G_1) + 1) + (c(G_2) + 2d(G_2) + 1) + (c(G_3) + 2d(G_3) + 1) = (c(G_1) + c(G_2) + c(G_3)) + 2(d(G_1) + d(G_2) + d(G_3) + 1) + 1 = c(G) + 2d(G) + 1$; the second equality holds by induction.

(3) $h(G) \leq 2c(G) + 3d(G) + 1$. We use induction on m. If $m = 1$, then the only internal vertex w of G is of type A, hence $h(G) = 0 < 1 = 2c(G) + 3d(G) + 1$. If $m > 1$, then the central vertex w of G is of one of types B–D.

Suppose first that w is of type C. Also, suppose that G_1 and G_2 have internal vertices; the other cases are analogous. Since w is of type C, we have $h(G) = h(G_1) + h(G_2)$, $c(G) = c(G_1) + c(G_2) + 1$, and $d(G) = d(G_1) + d(G_2)$. Hence, $h(G) = h(G_1) + h(G_2) \leq (2c(G_1) + 3d(G_1) + 1) + (2c(G_2) + 3d(G_2) + 1) = 2(c(G_1) + c(G_2) + 1) + 3(d(G_1) + d(G_2)) = 2c(G) + 3d(G) < 2c(G) + 3d(G) + 1$; the second inequality holds by induction.

Second, if w is of type D, we have $h(G) = h(G_1) + h(G_2) + h(G_3)$, $c(G) = c(G_1) + c(G_2) + c(G_3)$, and $d(G) = d(G_1) + d(G_2) + d(G_3) + 1$. Hence, $h(G) = h(G_1) + h(G_2) + h(G_3) \leq (2c(G_1) + 3d(G_1) + 1) + (2c(G_2) + 3d(G_2) + 1) + (2c(G_3) + 3d(G_3) + 1) = 2(c(G_1) + c(G_2) + c(G_3)) + 3(d(G_1) + d(G_2) + d(G_3) + 1) = 2c(G) + 3d(G) < 2c(G) + 3d(G) + 1$; the second inequality holds by induction.

Finally, suppose that w is of type B. Then $w_1 = w$ is the first vertex of a B-chain $w_1, \ldots, w_i$ of G. Recall that H is the only plane 3-tree child of w_i that has internal vertices. Let x be the central vertex of H. By the maximality of $w_1, \ldots, w_i$, we have that x is not of type B, hence x is of type A, C, or D. If x is of type A, we have $h(G) = 1$, $c(G) = d(G) = 0$, hence $h(G) = 1 = 2c(G) + 3d(G) + 1$.

If x is of type C, then let L_1 and L_2 be the children of H containing internal vertices. We have $h(G) = h(L_1) + h(L_2) + 1$, $c(G) = c(L_1) + c(L_2) + 1$, and $d(G) = d(L_1) + d(L_2)$. Thus, $h(G) = h(L_1) + h(L_2) + 1 \leq (2c(L_1) + 3d(L_1) + 1) + (2c(L_2) + 3d(L_2) + 1) + 1 =$

$2(c(L_1)+c(L_2)+1)+3(d(L_1)+d(L_2))+1 = 2c(G)+3d(G)+1$; the second inequality holds by induction.

Finally, if x is of type D, then let L_1, L_2, and L_3 be the children of H. We have $h(G) = h(L_1)+h(L_2)+h(L_3)+1$, $c(G) = c(L_1)+c(L_2)+c(L_3)$, and $d(G) = d(L_1)+d(L_2)+d(L_3)+1$. Thus, $h(G) = h(L_1)+h(L_2)+h(L_3)+1 \le (2c(L_1)+3d(L_1)+1)+(2c(L_2)+3d(L_2)+1)+(2c(L_3)+3d(L_3)+1)+1 = 2(c(L_1)+c(L_2)+c(L_3))+3(d(L_1)+d(L_2)+d(L_3)+1)+1 = 2c(G)+3d(G)+1$; the second inequality holds by induction.

(4) $x(G) \le b(G)$. This inequality follows from the fact that $x(G)$ is the number of vertices of type B of G none of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ passes through, hence this number cannot be larger than the number of vertices of type B of G.

(5) $x(G) \le 3h(G)$. Every internal vertex of G of type B belongs to a B-chain of G. Further, for every B-chain $w_1, w_2, \ldots, w_i$ of G, the curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ pass through all of $w_1, w_2, \ldots, w_i$, except for at most three vertices $u' = u_U$, $v' = v_V$, and $z' = z_Z$ (note that, in the description of the construction of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ if w is of type B, the vertices u, v, and z are not among $w_1, w_2, \ldots, w_i$). Thus, the number $x(G)$ of vertices of type B none of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ passes through is at most three times the number $h(G)$ of B-chains of G.

(6) $s(G) \ge 3a(G) + b(G) - x(G)$. We use induction on m. If $m = 1$ then the only internal vertex w of G is of type A, hence $a(G) = 1$ and $b(G) = x(G) = 0$. Further, by construction, each of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ passes through w, hence $s(G) = 3$. Thus, $s(G) = 3 = 3a(G)+b(G)-x(G)$. If $m > 1$, then the central vertex w of G is of one of types B–D.

Suppose first that w is of type C. Also, suppose that G_1 and G_2 have internal vertices; the other cases are analogous. Since w is of type C, we have $a(G) = a(G_1)+a(G_2)$, $b(G) = b(G_1) + b(G_2)$, and $x(G) = x(G_1) + x(G_2)$. By construction, the curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ contain all of $\lambda_u(G_1)$, $\lambda_v(G_1)$, $\lambda_w(G_1)$, $\lambda_u(G_2)$, $\lambda_z(G_2)$, and $\lambda_w(G_2)$. It follows that $s(G) = s(G_1)+s(G_2) \ge (3a(G_1)+b(G_1)-x(G_1))+(3a(G_2)+b(G_2)-x(G_2)) = 3(a(G_1) + a(G_2)) + (b(G_1) + b(G_2)) - (x(G_1) + x(G_2)) = 3a(G) + b(G) - x(G)$; the second inequality follows by induction.

Suppose next that w is of type D. Then we have $a(G) = a(G_1) + a(G_2) + a(G_3)$, $b(G) = b(G_1) + b(G_2) + b(G_3)$, and $x(G) = x(G_1) + x(G_2) + x(G_3)$. By construction, the curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ contain all of $\lambda_u(G_1)$, $\lambda_v(G_1)$, $\lambda_w(G_1)$, $\lambda_u(G_2)$, $\lambda_z(G_2)$, $\lambda_w(G_2)$, $\lambda_v(G_3)$, $\lambda_z(G_3)$, and $\lambda_w(G_3)$. It follows that $s(G) = s(G_1) + s(G_2) + s(G_3) \ge (3a(G_1)+b(G_1)-x(G_1))+(3a(G_2)+b(G_2)-x(G_2))+(3a(G_3)+b(G_3)-x(G_3)) = 3(a(G_1)+a(G_2)+a(G_3)) + (b(G_1)+b(G_2)+b(G_3)) - (x(G_1)+x(G_2)+x(G_3)) = 3a(G)+b(G)-x(G)$; the second inequality follows by induction.

Suppose finally that w is of type B. Then $w_1 = w$ is the first vertex of a B-chain $w_1, \ldots, w_i$ of G and H is the only plane 3-tree child of w_i that has internal vertices. Every internal vertex of G of type A is internal to H, hence $a(G) = a(H)$. Every internal vertex of G of type B is either an internal vertex of H of type B, or is one among $w_1, \ldots, w_i$; hence $b(G) = b(H)+i$. Since $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ contain all of $\lambda_{u'}(H)$, $\lambda_{v'}(H)$, and $\lambda_{z'}(H)$, we have that $s(G)$ is greater than or equal to $s(H)$ plus the number of vertices among

$w_1, \ldots, w_i$ through which the curves $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ pass. For the same reason, $x(G)$ is equal to $x(H)$ plus the number of vertices among $w_1, \ldots, w_i$ none of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ passes through. By construction, $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ do not pass through at most three vertices among $w_1, \ldots, w_i$, hence $x(G) \leq x(H)+3$ and $s(G) \geq s(H)+i-3$. Thus, we have $s(G) \geq s(H)+i-3 \geq 3a(H)+b(H)-x(H)+i-3 = 3a(H)+(b(H)+i)-(x(H)+3) \geq 3a(G)+b(G)-x(G)$; the second inequality follows by induction. $\qquad \square$

Lemma 3 can be used to prove that one of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ passes through many vertices of G. Let k be a parameter to be determined later.

If $a(G) \geq km$, then by (4) and (6) we get $s(G) \geq 3a(G) \geq 3km$.

If $a(G) < km$, by (1) and (6) we get $s(G) \geq 3a(G)+(m-a(G)-c(G)-d(G))-x(G)$, which by (5) becomes $s(G) \geq m+2a(G)-c(G)-d(G)-3h(G)$. Using (2) and (3) we get $s(G) \geq m+2(c(G)+2d(G)+1)-c(G)-d(G)-3(2c(G)+3d(G)+1) = m-5c(G)-6d(G)-1$. Again by (2) and by the assumption $a(G) < km$ we get $c(G)+2d(G)+1 < km$, thus $5c(G)+6d(G)+1 < 5c(G)+10d(G)+5 < 5km$. Hence, $s(G) \geq m-5km$.

Let $k = \frac{1}{8}$. We get $3km = m-5km = \frac{3m}{8}$, thus $s(G) \geq \frac{3m}{8}$ both if $a(G) \geq \frac{m}{8}$ and if $a(G) < \frac{m}{8}$. It follows that one of $\lambda_u(G)$, $\lambda_v(G)$, and $\lambda_z(G)$ is a proper good curve passing through $\lceil \frac{n-3}{8} \rceil$ internal vertices of G. This concludes the proof of Theorem 2.

We now prove that not only there is a planar straight-line drawing of a plane 3-tree with $\lceil \frac{n-3}{8} \rceil$ collinear vertices, but the geometric placement of these vertices can be arbitrarily prescribed, as long as it satisfies an ordering constraint. Since every plane graph of treewidth at most three is a subgraph of a plane 3-tree [17], this implies that every plane graph of treewidth at most three has a free collinear set with $\lceil \frac{n-3}{8} \rceil$ vertices.

Theorem 3. *Every collinear set in a plane 3-tree is also a free collinear set.*

Let G be an n-vertex plane 3-tree with external vertices u, v, and z in this counterclockwise order along the cycle (u, v, z). Consider any planar straight-line drawing Ψ of G and a horizontal line ℓ. Label each vertex of G as $\uparrow$, $\downarrow$, or $=$ according to whether it lies above, below, or on ℓ, respectively; let S be the set of vertices labeled $=$. Let E_ℓ be the set of edges of G that properly cross ℓ in Ψ; thus, the edges in E_ℓ have one end-vertex labeled $\uparrow$ and one end-vertex labeled $\downarrow$. Let $<_\Psi$ be the total ordering of $S \cup E_\ell$ corresponding to the left-to-right order in which the vertices in S and the crossing points between the edges in E_ℓ and ℓ appear along ℓ in Ψ.

Let X be any set of $|S|+|E_\ell|$ distinct points on ℓ. Each element in $S \cup E_\ell$ is associated with a point in X: The i-th element of $S \cup E_\ell$, where the elements in $S \cup E_\ell$ are ordered according to $<_\Psi$, is associated with the i-th point of X, where the points in X are in left-to-right order along ℓ. Let X_S and X_E be the subsets of the points in X associated to the vertices in S and to the edges in E_ℓ, respectively; also, let q_x be the point in X associated with a vertex $x \in S$ and by q_{xy} the point in X associated with an edge $(x, y) \in E_\ell$.

We have the following lemma, which implies Theorem 3.

Lemma 4. *There exists a planar straight-line drawing Γ of G such that:*

(1) Γ respects the labeling – *every vertex labeled* ↑*,* ↓*, or* = *is above, below, or on* ℓ*, respectively; and*

(2) Γ respects the ordering – *every vertex in* S *is placed at its associated point in* X_S *and every edge in* E_ℓ *crosses* ℓ *at its associated point in* X_E.

Proof. The proof is by induction on n and relies on a stronger inductive hypothesis, namely that the drawing Γ can be constructed even if the cycle (u, v, z) delimiting the outer face of G is represented in Γ by an arbitrarily prescribed triangle Δ satisfying the following properties:

(i) the vertices p_u, p_v, and p_z of Δ representing u, v, and z appear in this counter-clockwise order along Δ;

(ii) Δ respects the labeling – each of u, v, and z is above, below, or on ℓ if it has label ↑, ↓, or =, respectively; and

(iii) Δ respects the ordering – every vertex in $\{u, v, z\} \cap S$ lies at its associated point in X_S and every edge in $\{(u, v), (u, z), (v, z)\} \cap E_\ell$ crosses ℓ at its associated point in X_E.

In the base case $n = 3$. Let Δ be any planar straight-line drawing of cycle (u, v, z) satisfying properties (i)–(iii). Define Γ = Δ; then Γ is a planar straight-line drawing of G that respects the labeling and the ordering since Δ satisfies properties (i)–(iii).

Now assume that $n > 3$; let w be the central vertex of G, and let G_1, G_2, and G_3 be its children, whose outer faces are delimited by the cycles (u, v, w), (u, z, w), and (v, z, w), respectively. We distinguish some cases according to the labeling of u, v, z, and w. In every case we draw w at a point p_w and we draw straight-line segments from p_w to p_u, p_v, and p_z, obtaining triangles $Δ_1 = (p_u, p_v, p_w)$, $Δ_2 = (p_u, p_z, p_w)$, and $Δ_3 = (p_v, p_z, p_w)$. We then use induction to construct planar straight-line drawings of G_1, G_2, and G_3 in which the cycles (u, v, w), (u, z, w), and (v, z, w) delimiting their outer faces are represented by $Δ_1$, $Δ_2$, and $Δ_3$, respectively. Thus, we only need to ensure that each of $Δ_1$, $Δ_2$, and $Δ_3$ satisfies properties (i)–(iii). In particular, property (i) is satisfied as long as p_w is in the interior of Δ; property (ii) is satisfied as long as p_w respects the labeling; and property (iii) is satisfied as long as $p_w = q_w$, if $w \in S$, and each edge in $\{(u, w), (v, w), (z, w)\} \cap E_\ell$ crosses ℓ at its associated point, if $w \notin S$.

If all of u, v, and z have labels in the set $\{\uparrow, =\}$, then all the internal vertices of G have label ↑, by the planarity of Ψ, and the interior of Δ is above ℓ. Let p_w be any point in the interior of Δ (ensuring properties (i)–(ii) for $Δ_1$, $Δ_2$, and $Δ_3$). Also, $w \notin S$ and $(u, w), (v, w), (z, w) \notin E_\ell$, thus property (iii) is satisfied for $Δ_1$, $Δ_2$, and $Δ_3$.

The case in which all of u, v, and z have labels in the set $\{\downarrow, =\}$ is symmetric. If none of these cases applies, we can assume w.l.o.g. that u has label ↑ and v has label ↓.

• Suppose that z has label =. Since u has label ↑, v has label ↓, and (u, v, z) has this counter-clockwise orientation in G, the edge (u, v) and the vertex z are respectively the first and the last element in $S \cup E_\ell$ according to $<_\Psi$. Since Δ satisfies properties

(i)–(iii), the points q_{uv} and q_z are respectively the leftmost and the rightmost point in X; hence all the points in $X - \{q_{uv}, q_z\}$ are in the interior of Δ.
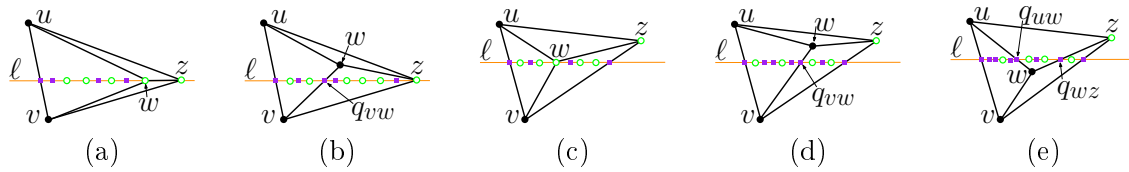

Figure 4: Cases for the proof of Lemma 4. The line ℓ is orange, the points in X_S are green, and the points in X_E are purple. (a) z and w have label $=$; (b) z has label $=$ and w has label $\uparrow$; (c) z has label $\uparrow$ and w has label $=$; (d) z and w have label $\uparrow$; and (e) z has label $\uparrow$ and w has label $\downarrow$.

- If w has label $=$, as in Fig. 4(a), then w is the last but one element in $S \cup E_\ell$ according to $<_\Psi$, by the planarity of Ψ (note that the edge (w, z) lies on ℓ). Since Δ satisfies (i)–(iii), the point q_w is the rightmost point in $X - \{q_z\}$. Let $p_w = q_w$ (ensuring properties (i)–(ii) for Δ_1, Δ_2, and Δ_3). Then w is at q_w and $(u, w), (v, w), (z, w) \notin E_\ell$ (ensuring property (iii) for Δ_1, Δ_2, and Δ_3).

- If w has label $\uparrow$, as in Fig. 4(b), then the edge (v, w) comes after the edge (u, v) and before the vertex z in $S \cup E_\ell$ according to $<_\Psi$, since (v, w) is an internal edge of G and Ψ is planar. Since Δ satisfies (i)–(iii), the point q_{vw} is between q_{uv} and q_z on ℓ. Draw a half-line h starting at v through q_{vw} and let p_w be any point in the interior of Δ (ensuring property (i) for Δ_1, Δ_2, and Δ_3) after q_{vw} on h (ensuring property (ii) for Δ_1, Δ_2, and Δ_3). Then $w \notin S$, $(u, w), (z, w) \notin E_\ell$, and the crossing point between (v, w) and ℓ is q_{vw} (ensuring property (iii) for Δ_1, Δ_2, and Δ_3).

- The case in which w has label $\downarrow$ is symmetric to the previous one.

• Assume now that z has label $\uparrow$. Since u and z have label $\uparrow$, since v has label $\downarrow$, and since (u, v, z) has this counter-clockwise orientation in G, the edges (u, v) and (v, z) are respectively the first and the last element in $S \cup E_\ell$ according to $<_\Psi$. Since Δ satisfies properties (i)–(iii), the points q_{uv} and q_{vz} are respectively the leftmost and the rightmost point in X; thus all the points in $X - \{q_{uv}, q_{vz}\}$ are in the interior of Δ.

- If w has label $=$, as in Fig. 4(c), then w comes after the edge (u, v) and before the edge (v, z) in $S \cup E_\ell$ according to $<_\Psi$, since w is an internal vertex of G and Ψ is planar. Since Δ satisfies (i)–(iii), q_w is between q_{uv} and q_{vz} on ℓ. Let $p_w = q_w$ (ensuring properties (i)–(ii) for Δ_1, Δ_2, and Δ_3). Then w is at q_w and $(u, w), (v, w), (z, w) \notin E_\ell$ (ensuring property (iii) for Δ_1, Δ_2, and Δ_3).

- If w has label $\uparrow$, as in Fig. 4(d), then the edge (v, w) comes after the edge (u, v) and before the edge (v, z) in $S \cup E_\ell$ according to $<_\Psi$, since (v, w) is an internal edge of G and Ψ is planar. Since Δ satisfies (i)–(iii), the point q_{vw} is between q_{uv} and q_{vz} on ℓ. Draw a half-line h starting at v through q_{vw} and let p_w be any

point in the interior of Δ (ensuring property (i) for Δ_1, Δ_2, and Δ_3) after q_{vw} on h (ensuring property (ii) for Δ_1, Δ_2, and Δ_3). Then $w \notin S$, $(u,w),(z,w) \notin E_\ell$, and the crossing point between (v,w) and ℓ is q_{vw} (ensuring property (iii) for Δ_1, Δ_2, and Δ_3).

- If w has label $\downarrow$, as in Fig. 4(e), then edges (u,v), (u,w), (w,z), and (v,z) come in this order in $S \cup E_\ell$ according to $<_\Psi$, since (u,w) and (w,z) are internal edges of G and Ψ is planar. Since Δ satisfies (i)–(iii), $q_{uv}, q_{uw}, q_{wz}, q_{vz}$ appear in this left-to-right order on ℓ. Let p_w be the intersection point between the line through u and q_{uw} and the line through z and q_{wz} (ensuring property (ii) for Δ_1, Δ_2, and Δ_3); note that p_w is in the interior of Δ (ensuring property (i) for Δ_1, Δ_2, and Δ_3). Then $w \notin S$, $(v,w) \notin E_\ell$, the crossing point between (v,w) and ℓ is q_{vw}, and the crossing point between (w,z) and ℓ is q_{wz} (ensuring property (iii) for Δ_1, Δ_2, and Δ_3).

- The case in which z has label $\downarrow$ is symmetric to the previous one.

  This concludes the proof of the lemma. □

5   Triconnected Cubic Planar Graphs

In this section we prove the following theorem.

Theorem 4. *Every n-vertex triconnected cubic plane graph admits a planar straight-line drawing with at least $\lceil \frac{n}{4} \rceil$ collinear vertices.*

By Theorem 1 it suffices to prove that, for every n-vertex triconnected cubic plane graph G, there exists a proper good curve λ passing through at least $\lceil \frac{n}{4} \rceil$ vertices of G.

Our proof will be by induction on n; Lemma 5 below states the inductive hypothesis satisfied by λ. In order to split the graph into subgraphs on which induction can be applied, we will use a structural decomposition that is derived from a paper by Chen and Yu [7], who proved that every n-vertex triconnected planar graph contains a cycle passing through $\Omega(n^{\log_3 2})$ vertices. This decomposition applies to a graph class, called *strong circuit graph* in [7], wider than triconnected cubic plane graphs. We will introduce the concept of *well-formed quadruple* in order to point out some properties of the graphs in this class. In particular, the inductive hypothesis will need to handle carefully a set (denoted by X below) of degree-2 vertices of the graph, which have neighbors that are not in the graph at the current level of the induction; since λ might pass through these neighbors, it has to avoid the vertices in X, in order to be good. Special conditions will be ensured for two vertices, denoted by u and v below, which work as a link to the rest of the graph.

We introduce some definitions. Consider a biconnected plane graph G. Given two external vertices u and v of G, let $\tau_{uv}(G)$ ($\beta_{uv}(G)$) be the path composed of the vertices and edges encountered when walking along the boundary of the outer face of G in clockwise (resp. counter-clockwise) direction from u to v. An *intersection point* between an open curve λ and $\beta_{uv}(G)$ (or $\tau_{uv}(G)$) is a point p belonging to both λ and $\beta_{uv}(G)$ (resp. $\tau_{uv}(G)$) such

that, for any $\epsilon > 0$, the part of λ in the disk centered at p with radius ϵ contains a point not in $\beta_{uv}(G)$ (resp. $\tau_{uv}(G)$). If the end-vertices of λ are in $\beta_{uv}(G)$ (or $\tau_{uv}(G)$), then we regard them as intersection points. An intersection point p between λ and $\beta_{uv}(G)$ (or $\tau_{uv}(G)$) is *proper* if, for any $\epsilon > 0$, the part of λ in the disk centered at p with radius ϵ contains a point in the outer face of G.

Our proof of the existence of a proper good curve passing through $\lceil \frac{n}{4} \rceil$ vertices of G is by induction on n. In order to make the induction work, we deal with the following setting. A quadruple (G, u, v, X) is *well-formed* if it satisfies the following properties.

(a) G is a biconnected subcubic plane graph;

(b) u and v are two distinct external vertices of G;

(c) $\delta_G(u) = \delta_G(v) = 2$;

(d) if the edge (u, v) exists, then it coincides with $\tau_{uv}(G)$;

(e) for every separation pair $\{a, b\}$ of G we have that a and b are external vertices of G and at least one of them is an internal vertex of $\beta_{uv}(G)$; further, every non-trivial $\{a, b\}$-component of G contains an external vertex of G different from a and b; and

(f) $X = (x_1, \dots, x_m)$ is a (possibly empty) sequence of degree-2 vertices of G in $\beta_{uv}(G)$, different from u and v, and in this order along $\beta_{uv}(G)$ from u to v.

We have the following main lemma (refer to Fig. 5).

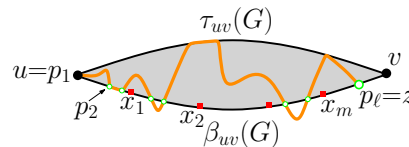
Figure 5: Illustration for the statement of Lemma 5. The gray region represents the interior of G. The curve λ is orange, the vertices in X are red squares, the intersection points between λ and $\beta_{uv}(G)$ are green circles, and the vertices u and v are black disks.

Lemma 5. *Let (G, u, v, X) be a well-formed quadruple. There exists a proper good curve λ such that:*

(1) λ starts at u, does not pass through v, and ends at a point z of $\beta_{uv}(G)$;

(2) z is in the part of $\beta_{uv}(G)$ between x_m and v (if $X = \emptyset$, this condition is vacuous);

(3) let $u = p_1, p_2, \dots, p_\ell = z$ be the intersection points between λ and $\beta_{uv}(G)$, ordered as they occur along λ; we have that $u = p_1, p_2, \dots, p_\ell = z, v$ appear in this order along $\beta_{uv}(G)$ (note that z is the "last" intersection between λ and $\beta_{uv}(G)$);

(4) λ passes through no vertex in X and all the vertices in X are incident to the unbounded region $R_{G,\lambda}$ of the plane defined by G and λ; in particular, if p_i, x_j and p_{i+1} come in this order along $\beta_{uv}(G)$, then the part of λ between p_i and p_{i+1} lies in the interior of G;

(5) λ and $\tau_{uv}(G)$ have no proper intersection point; and

(6) let L_λ and N_λ be the subsets of vertices in $V(G) - X$ the curve λ passes through and does not pass through, respectively; each vertex in N_λ can be charged to a vertex in L_λ so that each vertex in L_λ is charged with at most 3 vertices and u is charged with at most 1 vertex.

Before proceeding with the proof of Lemma 5, we state and prove an auxiliary lemma that will be used repeatedly in the remainder of the section.

Lemma 6. Let (G, u, v, X) be a well-formed quadruple and let $\{a, b\}$ be a separation pair of G with $a, b \in \beta_{uv}(G)$. The $\{a, b\}$-component G_{ab} of G containing $\beta_{ab}(G)$ either coincides with $\beta_{ab}(G)$ or consists of (see Fig. 6):

- a path $P_0 = (a, \dots, u_1)$ (possibly a single vertex $a = u_1$);

- for $i = 1, \dots, k$ with $k \geq 1$, a biconnected component G_i of G_{ab} that contains vertices u_i and v_i and such that (G_i, u_i, v_i, X_i) is a well-formed quadruple, with $X_i = X \cap V(G_i)$;

- for $i = 1, \dots, k - 1$, a path $P_i = (v_i, \dots, u_{i+1})$, where $u_{i+1} \neq v_i$; and

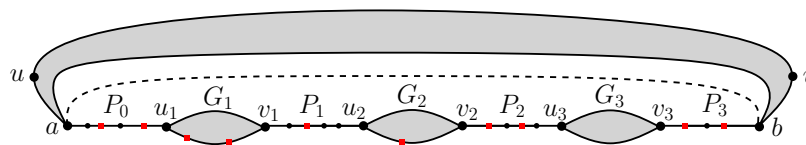- a path $P_k = (v_k, \dots, b)$ (possibly a single vertex $b = v_k$).

Figure 6: Illustration for Lemma 6 with $k = 3$.

Proof. If G contained more than two non-trivial $\{a, b\}$-components, then one of them would not contain any external vertex of G different from a and b, a contradiction to Property (e) of (G, u, v, X). Thus, G contains two non-trivial $\{a, b\}$-components, one of which is G_{ab}. Possibly, G contains a trivial $\{a, b\}$-component which is an internal edge (a, b) of G. The statement is proved by induction on the size of G_{ab}.

In the base case, G_{ab} is a path between a and b or is a biconnected graph. In the former case, G_{ab} coincides with $\beta_{ab}(G)$ and the statement of the lemma follows. In the latter case, the statement of the lemma follows with $k = 1$, $G_1 = G_{ab}$, $P_0 = a$, and $P_k = b$, as long as (G_{ab}, a, b, X_{ab}) is a well-formed quadruple, where $X_{ab} = X \cap V(G_{ab})$. We now prove that this is indeed the case.

- Property (a): G_{ab} is biconnected by hypothesis and subcubic since G is subcubic.

- Property (b): a and b are external vertices of G_{ab} as they are external vertices of G.

- Property (c): the degree of a and b in G_{ab} is at least 2, by the biconnectivity of G_{ab}, and at most 2, since G is subcubic and since a and b have a neighbor in the non-trivial $\{a, b\}$-component of G different from G_{ab}.

- Property (d): if the edge (a, b) exists in G, then it forms a trivial $\{a, b\}$-component and it does not belong to G_{ab}, hence the property is trivially satisfied.

- Property (e): consider any separation pair $\{a', b'\}$ of G_{ab}. If G_{ab} contained more than two non-trivial $\{a', b'\}$-components, as in Fig. 7(a), then one of them would be a non-trivial $\{a', b'\}$-component of G that contains no external vertex of G different from a' and b', a contradiction to Property (e) of (G, u, v, X). It follows that G_{ab} contains two non-trivial $\{a', b'\}$-components G'_{ab} and G''_{ab}.
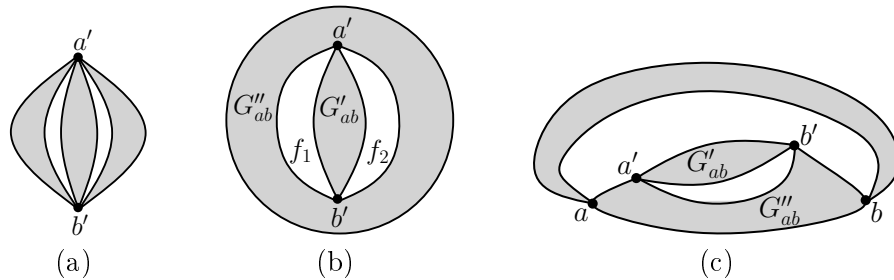

Figure 7: (a) G_{ab} contains more than two non-trivial $\{a', b'\}$-components. (b) G'_{ab} does not contain external vertices of G_{ab}. (c) a' and b' both belong to $\tau_{ab}(G_{ab})$.

There are at most two faces f_i of G_{ab}, with $i = 1, 2$, such that both G'_{ab} and G''_{ab} contain vertices different from a' and b' incident to f_i. If the outer face of G_{ab} was not one of f_1 and f_2, as in Fig. 7(b), then one of G'_{ab} and G''_{ab} would be a non-trivial $\{a', b'\}$-component of G that contains no external vertex of G different from a' and b', a contradiction to Property (e) of (G, u, v, X). It follows that both G'_{ab} and G''_{ab} contain external vertices of G_{ab} different from a' and b'; also, a' and b' are external vertices of G_{ab}. Now assume, for a contradiction, that a' and b' both belong to $\tau_{ab}(G_{ab})$, as in Fig. 7(c) (possibly $a' = a$, or $b' = b$, or both). Then a and b are both contained in the $\{a', b'\}$-component of G_{ab}, say G''_{ab}, containing $\beta_{ab}(G_{ab})$. It follows that G'_{ab} is a non-trivial $\{a', b'\}$-component of G containing no external vertex of G different from a' and b', a contradiction to Property (e) of (G, u, v, X). Hence, at least one of a' and b' is an internal vertex of $\beta_{ab}(G_{ab})$.

- Property (f): the vertices in X_{ab} have degree 2 in G_{ab} and are in $\beta_{ab}(G_{ab})$ since they have degree 2 in G and are in $\beta_{uv}(G)$. Note that $a, b \notin X$; indeed G_{ab} is biconnected and both a and b have neighbors not in G_{ab}, hence $\delta_G(a) = \delta_G(b) = 3$.

For the induction, we distinguish three cases.

In the first case a has a unique neighbor a' in G_{ab}. Then a' is an internal vertex of $\beta_{uv}(G)$. Since we are not in the base case, G_{ab} is not a simple path with two edges; hence, $\{a', b\}$ is a separation pair of G satisfying the conditions of the lemma. Let $G_{a'b}$ be the $\{a', b\}$-component of G containing $\beta_{a'b}(G)$. Then G_{ab} consists of $G_{a'b}$ together with the vertex a and the edge (a, a') and induction applies to $G_{a'b}$. If $G_{a'b}$ coincides with $\beta_{a'b}(G)$, then G_{ab} coincides with $\beta_{ab}(G)$, contradicting the fact that we are not in the base case. Hence, $G_{a'b}$ consists of: (i) a path $P_0' = (a', \ldots, u_1)$; (ii) for $i = 1, \ldots, k$ with $k \geq 1$, a biconnected component G_i of $G_{a'b}$ that contains vertices u_i and v_i and such that (G_i, u_i, v_i, X_i) is a well-formed quadruple; (iii) for $i = 1, \ldots, k-1$, a path $P_i = (v_i, \ldots, u_{i+1})$, where $u_{i+1} \neq v_i$; and (iv) a path $P_k = (v_k, \ldots, b)$. Then G_{ab} is composed of: (i) the path $(a, a') \cup P_0'$; (ii) for $i = 1, \ldots, k$, the biconnected component G_i of G_{ab}; (iii) for $i = 1, \ldots, k-1$, the path P_i; and (iv) the path P_k.

The second case, in which b has a unique neighbor in G_{ab}, is symmetric to the first one.

In the third case, the degree of both a and b in G_{ab} is greater than 1. Let G_1 be the biconnected component of G_{ab} containing a. Let H be the subgraph of G_{ab} induced by the vertices with at least one incident edge not in G_1. We prove the following claim: $b \notin V(G_1)$, and H and G_1 share a single vertex $a' \neq b$, which is an internal vertex of $\beta_{uv}(G)$.

Assume, for a contradiction, that $b \in V(G_1)$. Then G_{ab} is biconnected. Indeed, if G_1 contains a cut-vertex of G_{ab}, then this cut-vertex is also a cut-vertex of G, since $\{a, b\}$ is a separation pair of G and G_{ab} is an $\{a, b\}$-component of G; however, by Property (a) of (G, u, v, X) the graph G is biconnected. By the biconnectivity of G_{ab} and the maximality of G_1 we have $G_1 = G_{ab}$; hence, we are in the base case, a contradiction.

Every $G_1 \cup \{b\}$-bridge of G_{ab} has exactly one attachment in G_1. Indeed, if a $G_1 \cup \{b\}$-bridge B of G_{ab} had at least two attachments b_1 and b_2 in G_1, then there would exist a path Q between b_1 and b_2 in B, given that B is connected; however, the edges and the internal vertices of Q would not be in G_1, hence the union of B and Q would be a biconnected subgraph of G_{ab} containing G_1 as a proper subgraph, thus contradicting the maximality of G_1. An analogous proof shows that there is exactly one $G_1 \cup \{b\}$-bridge H. Let a' be the only attachment of H in G_1. Note that $\delta_H(a') = 1$, as $\delta_{G_1}(a') \geq 2$ since G_1 is biconnected. By the planarity of G, we have that a' is incident to the outer face of G_1, since a and b are both incident to the outer face of G. Since a' is the only attachment of H in G_1, it follows that a' is an internal vertex of $\beta_{uv}(G)$. This concludes the proof of the claim.

By the claim and since G_1 and H are not single edges, given that the degree of both a and b in G_{ab} is greater than 1, it follows that $\{a, a'\}$ and $\{a', b\}$ are separation pairs of G satisfying the statement of the lemma, hence induction applies to G_1 and H. In particular, (G_1, u_1, v_1, X_1) is a well-formed quadruple, with $X_1 = X \cap V(G_1)$, $u_1 = a$ and $v_1 = a'$. Further, H consists of: (i) for $i = 1, \ldots, k-1$ with $k \geq 2$, a path $P_i = (v_i, \ldots, u_{i+1})$ where $u_{i+1} \neq v_i$; note that $P_1 = (v_1 = a', \ldots, u_2)$ satisfies $u_2 \neq a'$ since $\delta_H(a') = 1$; (ii) for $i = 2, \ldots, k$, a biconnected component G_i of H containing vertices u_i and v_i (with $v_k = b$) and such that (G_i, u_i, v_i, X_i) is a well-formed quadruple, with $X_i = X \cap V(G_i)$. Then G_{ab} is composed of: (i) a path $P_0 = (a)$; (ii) for $i = 1, \ldots, k$ with $k \geq 1$, a biconnected component G_i that contains vertices u_i and v_i and such that (G_i, u_i, v_i, X_i) is a well-formed quadruple;

(iii) for $i = 1, \ldots, k-1$, a path $P_i = (v_i, \ldots, u_{i+1})$, where $u_{i+1} \neq v_i$; and (iv) a path $P_k = (b)$. This concludes the proof of the lemma. $\qquad\square$

We are now ready to prove Lemma 5. The proof is by induction on the size of G.

Base case: G is a simple cycle. Refer to Fig. 8. If u and v were not adjacent, then $\{u, v\}$ would be a separation pair none of whose vertices is internal to $\beta_{uv}(G)$, contradicting Property (e) of (G, u, v, X). Thus, the edge (u, v) exists and coincides with $\tau_{uv}(G)$ by Property (d). We now construct a proper good curve λ. The curve λ starts at u; it then passes through all the vertices in $V(G) - (X \cup \{v\})$ in the order in which they appear along $\beta_{uv}(G)$ from u to v; in particular, if two vertices in $V(G) - (X \cup \{v\})$ are consecutive in $\beta_{uv}(G)$, then λ contains the edge between them. If the neighbor v' of v in $\beta_{uv}(G)$ is not in X, then λ ends at v', otherwise λ ends at a point z in the interior of the edge (v, v'). Charge v to u and note that v is the only vertex in $V(G) - X$ that is not on λ. It is easy to see that λ is a proper good curve satisfying Properties (1)–(6).
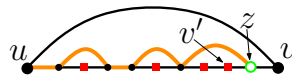

Figure 8: Base case for the proof of Lemma 5.

Next we describe the inductive cases. In the description of each inductive case, we implicitly assume that none of the previously described cases applies.

Case 1: The edge (u, v) exists. Refer to Fig. 9. By Property (d) of (G, u, v, X) the edge (u, v) coincides with $\tau_{uv}(G)$. By Property (c), the vertex v has two neighbors; let v' be the one different from u. Since G is not a simple cycle with length three, $\{u, v'\}$ is a separation pair of G to which Lemma 6 applies. If the $\{u, v'\}$-component of G containing $\beta_{uv'}(G)$ coincided with $\beta_{uv'}(G)$, then G would be a simple cycle, a contradiction to the fact that we are not in the base case. Hence, the graph G' obtained from G by removing the edge (u, v) consists of: (i) a path $P_0 = (u, \ldots, u_1)$; (ii) for $i = 1, \ldots, k$ with $k \geq 1$, a biconnected component G_i of G' that contains vertices u_i and v_i and such that (G_i, u_i, v_i, X_i) is a well-formed quadruple, where $X_i = X \cap V(G_i)$; (iii) for $i = 1, \ldots, k-1$, a path $P_i = (v_i, \ldots, u_{i+1})$, where $u_{i+1} \neq v_i$; and (iv) a path $P_k = (v_k, \ldots, v)$. Inductively construct a curve λ_i satisfying the properties of Lemma 5 for each quadruple (G_i, u_i, v_i, X_i). We construct a proper good curve λ for (G, u, v, X) as follows.
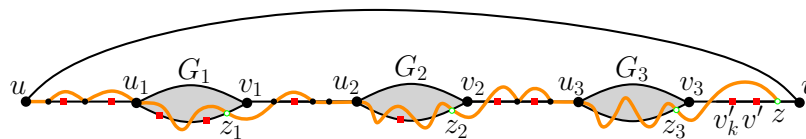

Figure 9: Case 1 of the proof of Lemma 5 with $k = 3$.

- The curve λ starts at u.

- It then passes through all the vertices in $V(P_0) \setminus X$ in the order as they appear along $\beta_{uv}(G)$ from u to u_1. Note that $u_1 \notin X$, since $\delta_G(u_1) = 3$, hence λ passes through u_1. This part of λ lies in the internal face of G incident to the edge (u, v).

- Suppose that λ has been constructed up to a vertex u_i, for some $1 \leq i \leq k$. Then λ contains λ_i, which terminates at a point z_i on $\beta_{u_i v_i}(G_i)$.

- Suppose that λ has been constructed up to a point z_i on $\beta_{u_i v_i}(G_i)$, for some $1 \leq i \leq k - 1$. Then λ continues with a curve in the outer face of G from z_i to the neighbor v_i' of v_i in P_i (if $v_i' \notin X$, as with $i = 1$ in Fig. 9) or from z_i to a point in the interior of the edge (v_i, v_i') (if $v_i' \in X$, as with $i = 2$ in Fig. 9).

- Suppose that λ has been constructed up to a point on the edge (v_i, v_i') (possibly coinciding with v_i'), for some $1 \leq i \leq k - 1$. Then λ passes through all the vertices in $V(P_i) \setminus (X \cup \{v_i\})$ in the order as they appear along $\beta_{uv}(G)$ from v_i to u_{i+1}. Note that $u_{i+1} \notin X$, since $\delta_G(u_{i+1}) = 3$, hence λ passes through u_{i+1}. This part of λ lies in the internal face of G incident to the edge (u, v).

- Finally, suppose that λ has been constructed up to a point z_k on $\beta_{u_k v_k}(G_k)$. If the neighbor v_k' of v_k in P_k is v, then λ terminates at z_k. Otherwise, λ continues with a curve in the outer face of G from z_k to v_k' (if $v_k' \notin X$) or from z_k to a point in the interior of the edge (v_k, v_k') (if $v_k' \in X$). Then λ passes through all the vertices in $V(P_k) \setminus (X \cup \{v_k, v\})$ in the order as they appear along $\beta_{uv}(G)$ from v_k to v. If $v' \in X$, then λ terminates at a point z along the edge (v', v), otherwise λ terminates at v'.

The curve λ satisfies Properties (1)–(5) of Lemma 5. In particular, the part of λ from z_i to a point on the edge (v_i, v_i') can be drawn without causing self-intersections because λ_i satisfies Properties (2), (3), and (5) by induction; in fact, these properties ensure that z_i and v_i' are both incident to R_{G,λ_i}. For $i = 1, \ldots, k$, the charge of the vertices in $N_\lambda \cap V(G_i)$ to the vertices in $L_\lambda \cap V(G_i)$ is determined inductively, thus each vertex in $L_\lambda \cap V(G_i)$ is charged with at most three vertices; charge v to u and observe that Property (6) is satisfied by the constructed charging scheme.

If Case 1 does not apply, then consider the graph $G' = G - \{v\}$. Since $\{u, v\}$ is not a separation pair of G, the vertex u is not a cut-vertex of G'. Let H be the biconnected component of G' containing u. We have the following claim.

Claim 1. The graph G has two $H \cup \{v\}$-bridges B_1 and B_2; further, each of B_1 and B_2 has two attachments, one of which is v; finally, one of B_1 and B_2 is an edge of $\tau_{uv}(G)$.

Proof. First, each $H \cup \{v\}$-bridge B_i of G has at most one attachment y_i in H, as otherwise B_i would contain a path (not passing through v) between two vertices of H, and H would not be maximal.

Second, if B_i had no attachment in H, then v would be a cut-vertex of G, whereas G is biconnected. Also, if v was not an attachment of B_i, then y_i would be a cut-vertex of G, whereas G is biconnected. Hence, B_i has two attachments, namely v and y_i. Further,

if there was a single $H \cup \{v\}$-bridge B_i, then y_i would be a cut-vertex of G, whereas G is biconnected. This and $\delta_G(v) = 2$ imply that G has two $H \cup \{v\}$-bridges B_1 and B_2.

Finally, one of y_1 and y_2, say y_1, belongs to $\tau_{uv}(G)$, while the other one, say y_2, belongs to $\beta_{uv}(G)$. Hence, if B_1 was not a trivial $H \cup \{v\}$-bridge, then $\{y_1, v\}$ would be a separation pair none of whose vertices is internal to $\beta_{uv}(G)$, which would contradict Property (e) of (G, u, v, X). This concludes the proof of the claim. □

By Claim 1 the graph G is composed of three subgraphs: a biconnected graph H, an edge $B_1 = (y_1, v)$, and a graph B_2, where H and B_1 share the vertex y_1, H and B_2 share the vertex y_2, and B_1 and B_2 share the vertex v. Before proceeding with the case distinction, we argue about the structure of H. Let $X' = \{y_2\} \cup (X \cap V(H))$. We have the following.

Claim 2. (H, u, y_1, X') is a well-formed quadruple.

Proof. Properties (a)–(c) are trivially satisfied by (H, u, y_1, X'). Concerning Property (d), if the edge (u, y_1) exists, then it is either $\tau_{uy_1}(H)$ or $\beta_{uy_1}(H)$, since $\delta_H(u) = 2$. However, $(u, y_1) \neq \beta_{uy_1}(H)$, since $y_2 \in \beta_{uy_1}(H)$ and $y_2 \notin \{u, y_1\}$.

Next, we discuss Property (e). Consider any separation pair $\{a, b\}$ of H. First, if a was not an external vertex of H, then $\{a, b\}$ would also be a separation pair of G such that a is not an external vertex of G; this would contradict Property (e) of (G, u, v, X). Second, if both a and b were in $\tau_{uy_1}(H)$, then $\{a, b\}$ would be a separation pair of G whose vertices are both in $\tau_{uv}(G)$, given that $\tau_{uy_1}(H) \subset \tau_{uv}(G)$; again, this would contradict Property (e) of (G, u, v, X). Third, if an $\{a, b\}$-component H_{ab} of H contained no external vertex of H different from a and b, then H_{ab} would also be an $\{a, b\}$-component of G containing no external vertex of G different from a and b, again contradicting Property (e) of (G, u, v, X).

Finally, we deal with Property (f). The vertices in $X \cap X'$ have degree 2 in H since they have degree 2 in G; further, they are internal to $\beta_{uy_1}(H)$ since they are internal to $\beta_{uv}(G)$. Further, we have that $\delta_H(y_2) = 2$. In order to prove the previous statement, note that $\delta_H(y_2) \geq 2$ since H is biconnected; further, $\delta_H(y_2) < \delta_G(y_2)$ since y_2 has a neighbor in B_2 not in H; finally, $\delta_G(y_2) \leq 3$ since G is a subcubic graph. Also, y_2 is an internal vertex of $\beta_{uy_1}(H)$, since it is an internal vertex of $\beta_{uv}(G)$ and is in H. This concludes the proof of the claim. □

Case 2: B_2 contains a vertex not in $X \cup \{v, y_2\}$. Refer to Fig. 10. The curve λ will be composed of three curves λ_1, λ_2, and λ_3.
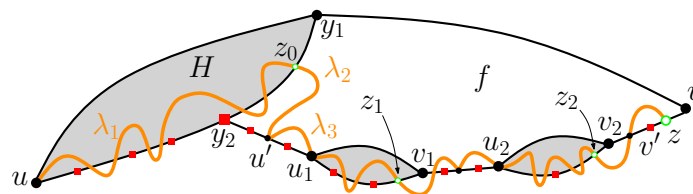

Figure 10: Case 2 of the proof of Lemma 5.

The curve λ starts at u. By Claim 2, a curve λ_1 satisfying the properties of Lemma 5 can be inductively constructed for (H, u, y_1, X'). Notice that $y_2 \in X'$, thus λ_1 terminates at a point z_0 in $\beta_{y_2 y_1}(H)$, by Property (2) of λ_1.

The curve λ_2 lies in the internal face f of G incident to the edge (y_1, v) and connects z_0 with a vertex u' in B_2 determined as follows. Traverse $\beta_{uv}(G)$ from y_2 to v and let $u' \neq y_2$ be the first encountered vertex not in X. By Property (f) of (G, u, v, X), every vertex in $X \cap V(B_2)$ has degree 2 in G and in B_2; also, $\delta_{B_2}(y_2) = \delta_{B_2}(v) = 1$. If all the internal vertices of $\beta_{y_2 v}(G)$ belong to X, then B_2 is a path whose internal vertices are in X, a contradiction to the hypothesis of Case 2. Hence, $u' \neq v$, $\beta_{y_2 u'}(G)$ is induced in B_2, u' is incident to f, and the interior of λ_2 crosses no edge of G. It is vital here that λ_1 satisfies Properties (3)–(5), ensuring that y_2 is not on λ_1 and that the edge incident to y_2 in B_2 is in R_{G, λ_1}. Thus, if such an edge is (y_2, u'), still λ intersects it only once.

The curve λ_3 connects u' with a point $z \notin \{y_2, v\}$ on $\beta_{y_2 v}(G)$. Note that $\{y_2, v\}$ is a separation pair of G, since by hypothesis B_2 is not an edge; further, y_2 and v both belong to $\beta_{uv}(G)$. Hence Lemma 6 applies and the curve λ_3 is constructed as in Case 1, with the part between y_2 and u' removed from the construction.

The curve λ satisfies Properties (1)–(5) of Lemma 5. We determine inductively the charge of the vertices in $(N_\lambda \cap V(H)) - \{y_2\}$ to the vertices in $L_\lambda \cap V(H)$, and the charge of the vertices in N_λ in each biconnected component G_i of B_2 to the vertices in $L_\lambda \cap V(G_i)$. The only vertices in N_λ that have not yet been charged to vertices in L_λ are y_2 and v; charge them to u'. Then u is charged with at most 1 vertex of H; every vertex in $L_\lambda - \{u, u'\}$ is charged with at most 3 vertices if it is in H or in a biconnected component of B_2, or with no vertex otherwise; finally, u' is charged with y_2, v, and with no other vertex if $\delta_G(u') = 2$ or with at most 1 other vertex if $\delta_G(u') = 3$; indeed, in the latter case $u' = u_1$ is such that induction is applied on a quadruple (G_1, u_1, v_1, X_1). Thus, Property (6) is satisfied by the constructed charging scheme.

If Case 2 does not apply, then B_2 is a path between y_2 and v whose internal vertices are in X. In order to proceed with the case distinction, we explore the structure of H.

Case 3: The edge (u, y_1) exists. By Claim 2, (H, u, y_1, X') is a well-formed quadruple, thus by Property (d) the edge (u, y_1) coincides with $\tau_{u y_1}(H)$. Let y' be the unique neighbor of y_1 in $\beta_{u y_1}(H)$.
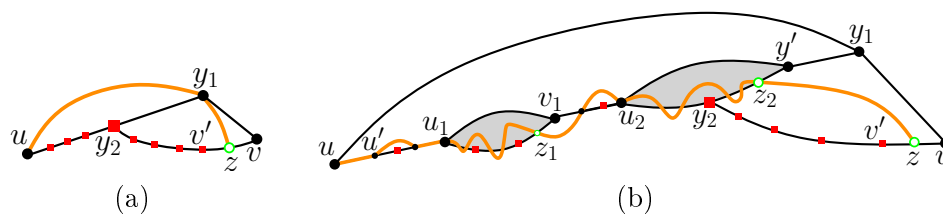

Figure 11: Case 3 of the proof of Lemma 5. (a) Every vertex of H different from u and y_1 is in X'. (b) H contains a vertex not in $X' \cup \{u, y_1\}$.

If every vertex of H different from u and y_1 is in X' (as in Fig. 11(a)), then λ consists

of the edge (u, y_1) together with a curve from y_1 to a point z along the edge (v, v'); the latter curve lies in the internal face of G incident to the edge (v, y_1). Charge y_2 and v to y_1 and note that λ satisfies Properties (1)–(6) required by Lemma 5.

If H contains a vertex not in $X' \cup \{u, y_1\}$ (as in Fig. 11(b)), then H contains at least 4 vertices; also, u and y' belong to $\beta_{uy_1}(H)$. Thus, Lemma 6 applies to separation pair $\{u, y'\}$ of H and a curve λ_1 can be constructed that connects u with a point $z_k \neq y_1$ on $\beta_{y_2 y_1}(H)$ as in Case 1. The curve λ consists of λ_1 and of a curve λ_2 lying in the internal face of G incident to the edge (v, y_1) and connecting z_k with a point z along the edge (v, v'). The curve λ satisfies Properties (1)–(5) of Lemma 5. We determine inductively the charge of the vertices in $N_\lambda - \{y_2\}$ in each biconnected component G_i of the graph obtained from H by removing the edge (u, y_1) to the vertices in $L_\lambda \cap V(G_i)$. We charge v to u, and y_1 and y_2 to the first vertex $u' \neq u$ not in X' encountered when traversing $\beta_{uy_1}(H)$ from u to y_1. It can be proved that u' exists, that $u' \neq y_1$, and that $u' \in L_\lambda$ as in Case 2, because of the assumption that H contains a vertex not in $X' \cup \{u, y_1\}$; then either zero or one vertex has been charged to u' so far, depending on whether $\delta_G(u') = 2$ or $\delta_G(u') = 3$, respectively, and Property (6) is satisfied by the constructed charging scheme.

If Case 3 does not apply, the consider the graph $H' = H - \{y_1\}$. Since we are not in Case 3, (u, y_1) is not an edge of H; also, by Claim 2 and Property (e) of (H, u, y_1, X'), $\{u, y_1\}$ is not a separation pair of H. It follows that u is not a cut-vertex of H'. Let K be the biconnected component of H' containing u. Analogously as in Claim 1, it can be proved that H has two $K \cup \{y_1\}$-bridges D_1 and D_2 such that D_1 is a trivial $K \cup \{y_1\}$-bridge (w_1, y_1) which is an edge of $\tau_{uy_1}(H)$ and such that D_2 has two attachments w_2 and y_1. We further distinguish the cases in which y_2 does or does not belong to K.

Case 4: $y_2 \in K$. Refer to Fig. 12. The vertices y_2 and w_2 are distinct. Indeed, if they were the same vertex, then $\delta_G(y_2) \geq 4$, as y_2 would have at least two neighbors in K, since K is biconnected, and one neighbor in each of B_2 and D_2; however, this would contradict the fact that G is a subcubic graph. Since $w_1, y_1 \in \tau_{uv}(G)$ and $y_2 \in \beta_{uv}(G)$, the vertices u, y_2, w_2, w_1 come in this order along $\beta_{uw_1}(K)$; it follows that D_2 is a trivial $K \cup \{y_1\}$-bridge, as otherwise $\{y_1, w_2\}$ would be a separation pair of G one of whose vertices is internal to G, while (G, u, v, X) is a well-formed quadruple.
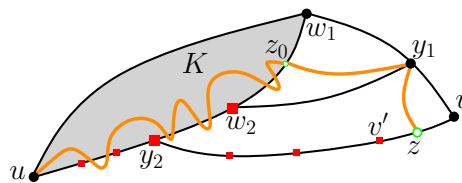

Figure 12: Case 4 of the proof of Lemma 5.

Let $X'' = (X \cap V(K)) \cup \{y_2, w_2\}$. Analogously as in Claim 2, it can be proved that (K, u, w_1, X'') is a well-formed quadruple. By induction, a curve λ_1 can be constructed satisfying the properties of Lemma 5 for (K, u, w_1, X''). In particular, λ_1 starts at u and ends at a point $z_0 \neq w_1$ in $\beta_{w_2 w_1}(K)$. The curve λ consists of λ_1, of a curve λ_2 from z_0 to y_1 lying in the internal face of G incident to the edge (w_1, y_1), and of a curve λ_3 from y_1 to a

point z along the edge (v, v') lying in the internal face of G incident to the edge (y_1, v). The curve λ satisfies Properties (1)–(5) of Lemma 5. Property (6) is satisfied by charging the vertices in $(N_\lambda \cap V(K)) - \{y_2, w_2\}$ to the vertices in $L_\lambda \cap V(K)$ as computed by induction, and by charging v, y_2, and w_2 to y_1.

Case 5: $y_2 \notin K$. Let $X'' = \{w_2\} \cup (X \cap V(K))$. It can be proved as in Claim 2 that (K, u, w_1, X'') is a well-formed quadruple. By induction, a curve λ_1 can be constructed satisfying the properties of Lemma 5 for (K, u, w_1, X''). In particular, λ_1 starts at u and ends at a point $z_0 \neq w_1$ in $\beta_{w_2 w_1}(K)$. The curve λ_1 is the first part of λ.

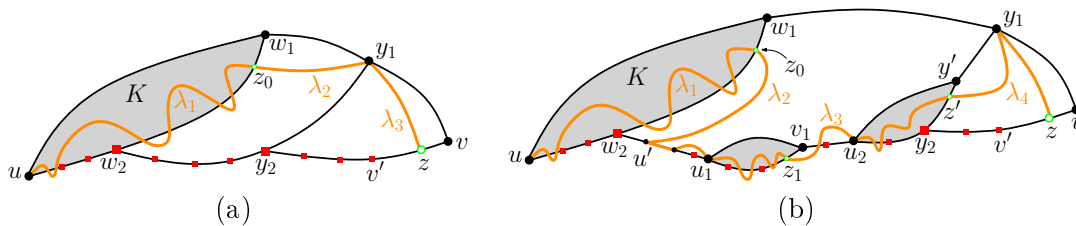(a)   (b)

Figure 13: Case 5 of the proof of Lemma 5. (a) Every vertex of D_2 different from w_2 and y_1 is in X'. (b) D_2 contains a vertex not in $X' \cup \{y_1, w_2\}$.

If every vertex of D_2 different from w_2 and y_1 is in X', as in Fig. 13(a), then λ continues with a curve λ_2 that connects z_0 with y_1 (λ_2 lies in the internal face of G incident to the edge (w_1, y_1)) and with a curve λ_3 that connects y_1 with a point z along the edge (v', v) (λ_3 lies in the internal face of G incident to the edge (y_1, v)).

If D_2 contains a vertex not in $X' \cup \{y_1, w_2\}$, as in Fig. 13(b), then, similarly to Case 2, λ continues with a curve λ_2 that connects z_0 with the first vertex $u' \neq w_2$ not in X' encountered while traversing $\beta_{w_2 y_1}(H)$ from w_2 to y_1; the curve λ_2 lies in the internal face of G incident to the edge (w_1, y_1). That u' exists, that $u' \neq y_1$, and that $u' \in L_\lambda$ can be proved as in Case 2 by the assumption that D_2 contains a vertex not in $X' \cup \{y_1, w_2\}$. Then λ continues with a curve λ_3 that connects u' with a point z' in $\beta_{y_2 y_1}(H)$; as in Case 2, $\{w_2, y_1\}$ is a separation pair of H, hence Lemma 6 applies and the curve λ_3 is constructed as in Case 1, with the part between w_2 and u' removed from the construction. Finally, if z' is not a point internal to the edge (y', y_1), where y' is the unique neighbor of y_1 in $\beta_{u y_1}(H)$, the curve λ contains a curve λ_4 that connects z' with y_1, and then y_1 with a point z on the edge (v, v'); the curve λ_4 lies in the internal face of G incident to the edge (y_1, v). Otherwise, we redraw the last part of λ_3 so that it terminates at y_1 rather than at z'; we then let λ_4 connect y_1 with a point z on the edge (v, v') in the internal face of G incident to the edge (y_1, v).

The curve λ satisfies Properties (1)–(5) of Lemma 5. We determine inductively the charge of the vertices in $(N_\lambda \cap V(K)) - \{w_2\}$ to the vertices in $L_\lambda \cap V(K)$, as well as the charge of the vertices in $N_\lambda - \{y_2\}$ in each biconnected component G_i of D_2, if any, to the vertices in $L_\lambda \cap V(G_i)$. Charge v, y_2, and w_2 to y_1. Property (6) is satisfied by the constructed charging scheme. This concludes the proof of Lemma 5.

We now apply Lemma 5 to prove Theorem 4. Let G be any triconnected cubic plane

graph. Let G' be the plane graph obtained from G by removing any edge (u, v) incident to the outer face of G, where u is encountered right before v when walking in clockwise direction along the outer face of G. Let $X' = \emptyset$. We have the following.

Lemma 7. (G', u, v, X') *is a well-formed quadruple.*

Proof. Concerning Property (a) G' is a subcubic plane graph since G is. Also, G' is bi-connected, since G is triconnected. Concerning Property (b), the vertices u and v are external vertices of G' since they are external vertices of G. Concerning Property (c), $\delta_{G'}(u) = \delta_{G'}(v) = 2$ since $\delta_G(u) = \delta_G(v) = 3$. Properties (d) and (f) are trivially satisfied since the edge (u, v) does not belong to G' and since $X' = \emptyset$, respectively.

We now prove Property (e). Consider any separation pair $\{a, b\}$ of G'. If G' had at least 3 non-trivial $\{a, b\}$-components, then G would have at least 2 non-trivial $\{a, b\}$-components, whereas it is triconnected. Hence, G' has 2 non-trivial $\{a, b\}$-components H and H'. The vertices u and v are not in the same non-trivial $\{a, b\}$-component of G', as otherwise G would not be triconnected. This implies that $\{a, b\} \cap \{u, v\} = \emptyset$. Both H and H' contain external vertices of G' (in fact u and v). It follows that a and b are both external vertices of G'. Hence, the vertices u, a, v, and b come in this order along the boundary of the outer face of G', thus one of a and b is internal to $\tau_{uv}(G')$, while the other one is internal to $\beta_{uv}(G')$. This concludes the proof of the lemma. $\square$

It follows by Lemma 7 that a proper good curve λ can be constructed satisfying the properties of Lemma 5. Insert the edge (u, v) in the outer face of G', restoring the plane embedding of G. By Properties (1)–(5) of λ this insertion can be accomplished so that (u, v) does not intersect λ other than at u, hence λ remains proper and good. In particular, the end-points u and z of λ both belong to $\beta_{uv}(G')$, while the insertion of (u, v) only prevents the internal vertices of $\tau_{uv}(G')$ from being incident to $R_{G,\lambda}$. By Property (6) of λ with $X' = \emptyset$, each vertex in N_λ is charged to a vertex in L_λ, and each vertex in L_λ is charged with at most three vertices in N_λ. Thus, λ is a proper good curve passing through $\lceil \frac{n}{4} \rceil$ vertices of G. This concludes the proof of Theorem 4.

6 Planar Graphs with Large Tree-width

In this section we prove the following theorem.

Theorem 5. *Let G be a planar graph and k be its tree-width. There exists a planar straight-line drawing of G with $\Omega(k^2)$ collinear vertices.*

Let G be a planar graph with tree-width k. We assume that G is connected; indeed, if it were not, then it would contain a connected component with tree-width k and in order to prove Theorem 5 it would suffice to construct a planar straight-line drawing of that connected component with $\Omega(k^2)$ collinear vertices. In order to prove that G admits a planar straight-line drawing with $\Omega(k^2)$ collinear vertices we exploit Theorem 1, as well as a result of Robertson, Seymour and Thomas [23], which asserts that G contains a $g \times g$ grid H as a minor, where g is at least $(k + 4)/6$.

Let $v_{i,j}$ be the vertices of H, with $1 \leq i,j \leq g$, where $v_{i,j}$ and $v_{i',j'}$ are adjacent in H if and only if $|i-i'|+|j-j'| = 1$. Let $G_{i,j}$ be the connected subgraph of G represented by $v_{i,j}$ in H. Note that, since H is a minor of G, each vertex $v_{i,j}$ of H corresponds to a connected subgraph $G_{i,j}$ of G such that, starting from G and contracting all the edges in the graphs $G_{i,j}$, the graph H is obtained; then $v_{i,j}$ is the vertex obtained by contracting all the edges of $G_{i,j}$. By the planarity of G, every edge of G that is incident to a vertex in $G_{i,j}$, for some $2 \leq i,j \leq g-1$, has its other end-vertex in a graph $G_{i',j'}$ such that $|i-i'| \leq 1$ and $|j-j'| \leq 1$. (The previous statement might not be true for an edge that is incident to a vertex in $G_{i,j}$ with $i=1$, $i=g$, $j=1$, or $j=g$.)

Refer to Fig. 14(a). For every edge $(v_{i,j}, v_{i+1,j})$ of H, arbitrarily choose an edge $e_{i,j}$ connecting a vertex in $G_{i,j}$ and a vertex in $G_{i+1,j}$ as the *reference edge* for the edge $(v_{i,j}, v_{i+1,j})$ of H. Such an edge exists since H is a minor of G. Reference edges $e'_{i,j}$ for the edges $(v_{i,j}, v_{i,j+1})$ of H are defined analogously.
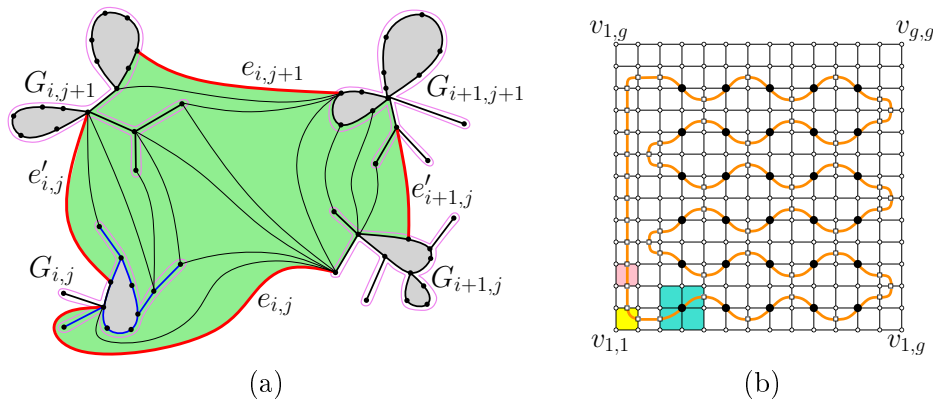

(a)   (b)

Figure 14: (a) Cells, boundaries, and references edges. The cell $C_{i,j}$ is green. The graphs $G_{i,j}$, $G_{i+1,j}$, $G_{i,j+1}$, and $G_{i+1,j+1}$ are surrounded by violet curves; their interior is gray. The references edges are red and thick. The right-top boundary of $G_{i,j}$ is blue. (b) The construction of λ (represented as a thick orange line). Large disks represent the graphs $G_{i,j}$ such that λ passes through vertices of $G_{i,j}$. Small circles represent the graphs $G_{i,j}$ such that λ does not pass through any vertex of $G_{i,j}$. White squares represent the intersections between λ and reference edges.

For every pair of indices $1 \leq i,j \leq g-1$, we call *right-top* boundary of $G_{i,j}$ the walk that starts at the end-vertex of $e'_{i,j}$ in $G_{i,j}$, traverses the boundary of the outer face of $G_{i,j}$ in clockwise direction and ends at the end-vertex of $e_{i,j}$ in $G_{i,j}$. The *right-bottom* boundary of $G_{i,j}$ (for every $1 \leq i \leq g-1$ and $2 \leq j \leq g$), the *left-top* boundary of $G_{i,j}$ (for every $2 \leq i \leq g$ and $1 \leq j \leq g-1$), and the *left-bottom* boundary of $G_{i,j}$ (for every $2 \leq i,j \leq g$) are defined analogously.

For each $1 \leq i,j \leq g-1$, we define the *cell* $C_{i,j}$ as the bounded closed region of the plane that is delimited by (in clockwise order along the boundary of the region): the right-top boundary of $G_{i,j}$, the edge $e'_{i,j}$, the right-bottom boundary of $G_{i,j+1}$, the edge $e_{i,j+1}$, the left-bottom boundary of $G_{i+1,j+1}$, the edge $e'_{i+1,j}$, the left-top boundary of $G_{i+1,j}$, and

the edge $e_{i,j}$.

We construct a proper good curve passing through $\Omega(g^2) \in \Omega(k^2)$ vertices of G. For the sake of the simplicity of the description, we construct a *closed* curve λ passing through $\Omega(g^2)$ vertices of G and such that, for each edge e of G, either λ contains e or λ has at most one point in common with e. Then λ can be turned into a proper good curve by cutting off a piece of it in the interior of an internal face f of G and by changing the outer face of G to f.

The curve λ passes through (at least) one vertex of each graph $G_{i,j}$ with i and j even, and with $4 \leq i \leq g'$ and $2 \leq j \leq g'$, where g' is the largest integer divisible by 4 and smaller than or equal to $g - 2$; note that there are $\Omega(g^2) \in \Omega(k^2)$ such graphs $G_{i,j}$. Then Theorem 5 follows from Theorem 1. The curve λ is composed of several good curves, each one connecting two points in the interior of two reference edges for edges of H. Refer to Fig. 14(b). In particular, each open curve is of one of the following types.

- *Type A: Cell traversal curve.* A curve γ connecting two points $p(\gamma)$ and $q(\gamma)$ in the interior of reference edges $e_{i,j}$ and $e_{i,j+1}$, or of reference edges $e'_{i,j}$ and $e'_{i+1,j}$. See, e.g., the part of λ in the pink region in Fig. 14(b).

- *Type B: Cell turn curve.* A curve γ connecting two points $p(\gamma)$ and $q(\gamma)$ in the interior of reference edges $e_{i,j}$ and $e'_{i,j}$, or of reference edges $e'_{i,j}$ and $e_{i,j+1}$, or of reference edges $e_{i,j+1}$ and $e'_{i+1,j}$, or of reference edges $e'_{i+1,j}$ and $e_{i,j}$. See, e.g., the part of λ in the yellow region in Fig. 14(b).

- *Type C: Vertex getter curve.* A curve γ connecting two points $p(\gamma)$ and $q(\gamma)$ in the interior of reference edges $e'_{i,j-1}$ and $e'_{i+2,j}$ or of reference edges $e'_{i,j}$ and $e'_{i+2,j-1}$, and passing through a vertex of $G_{i+1,j}$. See, e.g., the part of λ in the turquoise region in Fig. 14(b).

To each open curve γ composing λ we associate a distinct region $R(\gamma)$ of the plane, so that γ lies in $R(\gamma)$. For curves γ of Type A or B, the region $R(\gamma)$ is the unique cell delimited by the reference edges containing $p(\gamma)$ and $q(\gamma)$. For a curve γ of Type C, the region $R(\gamma)$ consists of the interior of $G_{i+1,j}$ together with the four cells incident to the boundary of $G_{i+1,j}$.

Any two regions associated to distinct open curves do not intersect, except along their boundaries. Further, for every region $R(\gamma)$ and for every edge e of G, either e is in $R(\gamma)$ or it has no intersection with the interior of $R(\gamma)$. Thus, in order to prove that λ has at most one point in common with every edge of G, it suffices to show how to draw γ so that it lies in the interior of $R(\gamma)$, except at points $p(\gamma)$ and $q(\gamma)$, and so that it has at most one common point with each edge in the interior of $R(\gamma)$. In order to describe how to draw γ, we distinguish the cases in which γ is of Type A, B, or C.

If γ is of Type A or B (refer to Fig. 15(a)), draw the dual graph D of G so that each edge of D only intersects its dual edge; restrict D to the vertices and edges in the interior of $R(\gamma)$, obtaining a graph D^*; find a simple path P in D^* connecting the vertices f_p and f_q of D^* incident to the reference edges to which $p(\gamma)$ and $q(\gamma)$ belong (note that P exists

since the region of the plane defined by each cell is connected and hence so is D^*); draw γ as P plus two curves connecting f_p and f_q with $p(\gamma)$ and $q(\gamma)$, respectively. Also, γ intersects each edge of G at most once, since P does. Finally, γ lies in the interior of $R(\gamma)$, except at points $p(\gamma)$ and $q(\gamma)$. Thus, γ satisfies the required properties.
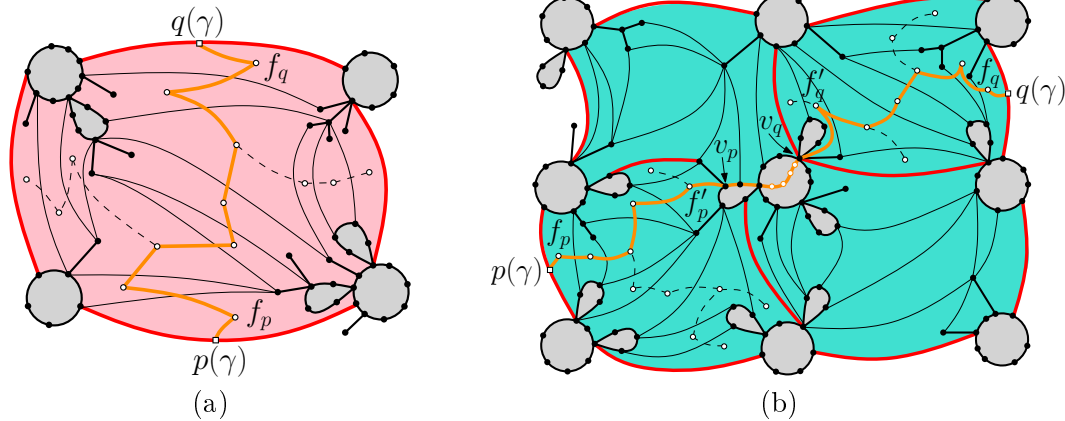

Figure 15: (a) Drawing a curve γ of Type A. The region $R(\gamma)$ is pink. The graph D^* has vertices represented by white circles; the edges of D^* in P are thick orange lines, while the edges of D^* not in P are dashed black lines. (b) Drawing a curve γ of Type C. The region $R(\gamma)$ is turquoise. The internal vertices of the path P in $G_{i+1,j}$ are black disks if they belong to the boundary of $G_{i+1,j}$, or orange circles if they are internal vertices of $G_{i+1,j}$.

If γ is of Type C (refer to Fig. 15(b)), assume that γ connects two points $p(\gamma)$ and $q(\gamma)$ respectively belonging to the interior of $e'_{i,j-1}$ and $e'_{i+2,j}$; the case in which $p(\gamma)$ and $q(\gamma)$ respectively belong to the interior of $e'_{i,j}$ and $e'_{i+2,j-1}$ is analogous. The curve γ is composed of three curves, namely: (1) a curve γ_1 that connects $p(\gamma)$ and a vertex v_p on the left-bottom boundary of $G_{i+1,j}$, and that lies in the interior of $C_{i,j-1}$, except at $p(\gamma)$ and v_p; (2) a curve γ_2 that connects v_p and a vertex v_q on the right-top boundary of $G_{i+1,j}$, and that is an induced path in $G_{i+1,j}$; and (3) a curve γ_3 that connects v_q and $q(\gamma)$, and that lies in the interior of $C_{i+1,j}$, except at v_q and $q(\gamma)$. The curve γ_2 might degenerate to be a single point if $v_p = v_q$.

We start with γ_2. Consider a path P in $G_{i+1,j}$ which is a shortest path connecting a vertex on the left-bottom boundary of $G_{i+1,j}$ and a vertex on the right-top boundary of $G_{i+1,j}$. Let v_p and v_q be the end-vertices of P. Note that, possibly, $v_p = v_q$. Such a path P always exists since $G_{i+1,j}$ is connected; also, P has no internal vertex incident to the left-bottom boundary or to the right-top boundary of $G_{i+1,j}$, as otherwise there would exist a path shorter than P between a vertex on the left-bottom boundary of $G_{i+1,j}$ and a vertex on the right-top boundary of $G_{i+1,j}$. Draw γ_2 as P.

In order to draw γ_1 (the curve γ_3 is drawn similarly), draw the dual graph D of G so that each edge of D only intersects its dual edge; restrict D to the vertices and edges in the interior of $C_{i,j-1}$, obtaining a graph D^*; find a shortest path P_p in D^* connecting the vertex f_p of D^* incident to the reference edge to which $p(\gamma)$ belongs and a vertex representing a

face of G incident to v_p. Let f'_p be the second end-vertex of such a path; draw γ_1 as P plus two curves connecting f_p and f'_p with $p(\gamma)$ and v_p, respectively.

The curve γ has no intersections with the boundary of $R(\gamma)$ other than at $p(\gamma)$ and $q(\gamma)$. We now prove that γ intersects each edge in $R(\gamma)$ at most once. First, γ intersects each edge of $G_{i+1,j}$ at most once, since γ_2 is a shortest path in $G_{i+1,j}$ and since γ_1 and γ_3 have no intersections with the edges of $G_{i+1,j}$, except at v_p and v_q. Second, γ intersects each edge in $C_{i,j-1}$ at most once, since P_p does, since γ_1 does not cross any edge incident to v_p (given that P_p is a shortest path between f_p and any face incident to v_p), and since γ_2 and γ_3 do not intersect edges in $C_{i,j-1}$ other than at v_p (given that P does not contain any vertex incident to the left-bottom boundary of $G_{i+1,j}$ other than v_p); similarly, γ intersects each edge in $C_{i+1,j}$ at most once. Third, γ intersects each edge in $C_{i+1,j-1}$ at most once, namely at its possible end-vertex in $G_{i+1,j}$; similarly, γ intersects each edge in $C_{i,j}$ at most once. Thus, γ satisfies the required properties.

This concludes the proof of Theorem 5.

7 Implications for other graph drawing problems

In this section, we present a number of corollaries of our results to other graph drawing problems. The following lemma is one of the key tools to establish these connections. For the sake of completeness we explicitly state it here (in a more readily applicable form than the original, see [4, Lemma 1]).

Lemma 8. *[4] Let G be a planar graph that has a planar straight-line drawing Γ in which a (collinear) set $S \subseteq V(G)$ of vertices lie on the x-axis. Then, for an arbitrary assignment of y-coordinates to the vertices in S, there exists a planar straight-line drawing Γ' of G such that each vertex in S has the same x-coordinate as in Γ and has the assigned y-coordinate.*

The above lemma immediately implies the following.

Lemma 9. *[4] Let G be a planar graph, $R \subseteq V(G)$ be a free collinear set, and $<_R$ be the total order associated with R. Consider any assignment of x- and y-coordinates to the vertices in R such that the assigned x-coordinates are all distinct and the order by increasing x-coordinates of the vertices in R is $<_R$ (or its reversal). Then there exists a planar straight-line drawing of G such that each vertex in R has the assigned x- and y-coordinates.*

We first apply Lemma 9 to obtain an optimal bound (up to a multiplicative constant) on the size of universal point subsets for planar graphs of treewidth at most three.

Corollary 1. *Every set P of at most $\lceil \frac{n-3}{8} \rceil$ points in the plane is a universal point subset for all n-vertex plane graphs of treewidth at most three.*

Proof. If necessary, rotate the Cartesian axes so that no two points in P have the same x-coordinate. By Theorems 2 and 3 every n-vertex plane graph G of treewidth at most three has a free collinear set R of cardinality $|P|$. Let $<_R$ be the total order associated with R. Since no two points in P have the same x-coordinate, there exists a bijective mapping

$\delta : R \to P$ such that, for every two vertices $v, w \in R$, $v <_R w$ if and only if the x-coordinate of point $\delta(v)$ is smaller than the x-coordinate of point $\delta(w)$. Then by Lemma 9 there exists a planar straight-line drawing of G that respects the mapping δ. □

It is implicit in [4] and explicit in [21] (in both cases using Lemmata 8 and 9 above), that every straight-line drawing (possibly with crossings) of a planar graph G can be untangled while keeping at least $\sqrt{\chi}$ vertices fixed, where χ is the size of a free collinear set of G. Together with Theorems 2 and 3 this implies the following corollary.

Corollary 2. *Any straight-line drawing (possibly with crossings) of an n-vertex planar graph of treewidth at most three can be untangled while keeping at least $\sqrt{\lceil (n-3)/8 \rceil}$ vertices fixed.*

We conclude this section with the application to column planar sets. Lemma 8 implies that every collinear set is a column planar set. That and our three main results imply our final corollary.

Corollary 3. *(a) Triconnected cubic planar graphs have column planar sets of linear size. (b) Planar graphs of treewidth at most three have column planar sets of linear size. (c) Planar graphs of treewidth at least k have column planar sets of size $\Omega(k^2)$.*

8 Conclusions

In this paper we studied the problem of constructing planar straight-line graph drawings with many collinear vertices. It would be interesting to tighten the best known bounds (which are $\Omega(n^{0.5})$ and $O(n^{0.986})$) for the maximum number of vertices that can be made collinear in a planar straight-line drawing of any n-vertex planar graph. In particular, we ask: Is it true that, if a plane graph G has a dual graph that contains a cycle with m vertices, then G has a planar straight-line drawing with $\Omega(m)$ collinear vertices? A positive answer to this question would improve the $\Omega(n^{0.5})$ lower bound to $\Omega(n^{0.694})$ (via the result in [15]). As noted in the Introduction, the "converse" is true for maximal plane graphs: If a maximal plane graph G has a planar straight-line drawing with χ collinear vertices, then the dual graph D of G has a cycle with $\Omega(\chi)$ vertices.

We proved that every n-vertex triconnected cubic plane graph has a planar straight-line drawing with $\lceil \frac{n}{4} \rceil$ collinear vertices. It seems plausible that an $\Omega(n)$ lower bound holds true for every n-vertex subcubic plane graph. Recall from the Introduction that the linear lower bound does not extend to all bounded-degree planar graphs [18], in fact, it does not extend already to all planar graphs of maximum degree 7.

We proved that n-vertex plane graphs with treewidth at most three have planar straight-line drawings with $\lceil \frac{n-3}{8} \rceil$ collinear vertices. Of our three results, this one has the widest applications to other graph drawing problems due to the fact that gives a free collinear set of size $\lceil \frac{n-3}{8} \rceil$. In fact, we proved that every collinear set is a free collinear set in planar 3-trees. This brings us to an open question already posed by Ravsky and Verbitsky [21]: is every collinear set a free collinear set, and if not, how close are the sizes of these two sets in a general planar graph?

Finally, we can prove that the maximum number of collinear vertices in any planar straight-line drawing of a plane 3-tree G can be computed in polynomial time (the statement extends to a *planar* 3-tree by choosing the outer face in every possible way). Indeed, there are six (topologically distinct) ways in which a proper good curve λ can "cut" the 3-cycle C delimiting the outer face of G: in three of them λ passes through a vertex of C and properly crosses the edge of C not incident to that vertex, and in the other three λ properly crosses two edges of C. This associates to G six parameters, representing the maximum number of internal vertices of G through which these six curves can pass. Further, the six parameters for G can be easily computed as a function of the same parameters for the plane 3-trees children of G. This leads to a polynomial-time dynamic-programming algorithm to compute the six parameters and consequently the maximum number of collinear vertices in any planar straight-line drawing of G. By implementing this idea, we have observed the following fact: For every $m \leq 50$ and for every plane 3-tree G with m internal vertices, there exists a planar straight-line drawing of G with $\lceil \frac{m+2}{3} \rceil$ collinear internal vertices (and this bound is the best possible for all $m \leq 50$). It would be interesting to prove that this is the case for every $m \geq 1$.

Acknowledgments

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