FORCING SUBARRANGEMENTS IN COMPLETE ARRANGEMENTS OF PSEUDOCIRCLES

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ABSTRACT. In arrangements of pseudocircles (i.e., Jordan curves) the weight of a vertex (i.e., an intersection point) is the number of pseudocircles that contain the vertex in its interior. We show that in complete arrangements (in which each two pseudocircles intersect) $2n - 1$ vertices of weight 0 force an $\alpha$-subarrangement, a certain arrangement of three pseudocircles. Similarly, $4n - 5$ vertices of weight 0 force an $\alpha^4$-subarrangement (of four pseudocircles). These results on the one hand give improved bounds on the number of vertices of weight $\leq k$ for complete, $\alpha$-free and complete, $\alpha^4$-free arrangements. On the other hand, interpreting $\alpha$- and $\alpha^4$-arrangements as complete graphs with three and four vertices, respectively, the bounds correspond to known results in extremal graph theory.

1 Introduction

An arrangement of pseudocircles is a finite set $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ of simple closed curves (pseudocircles) in the plane such that

(i) no three curves meet each other at the same point,

(ii) each two curves $\gamma_i, \gamma_j$ have at most two points in common, and

(iii) these intersection points in $\gamma_i \cap \gamma_j$ are always points where $\gamma_i, \gamma_j$ cross each other.

An arrangement is complete if each two pseudocircles intersect.

Any arrangement induces a planar embedding of a graph whose vertices are the intersection points between the pseudocircles and whose edges are the curves between these intersections. In the following, we will often refer to this arrangement graph when talking about vertices, edges, and faces of the arrangement.

Definition 1.1. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be an arrangement of pseudocircles. The weight of a vertex $V$ is the number of pseudocircles $\gamma_i$ such that $V$ is contained in $\text{int}(\gamma_i)$, the interior of $\gamma_i$. Weights of edges and faces are defined accordingly.

We will consider the number $v_k = v_k(\Gamma)$ of vertices of given weight $k$, the number $v_{\leq k} = v_{\leq k}(\Gamma)$ of vertices of weight $\leq k$, and the number $v_{\geq k} = v_{\geq k}(\Gamma)$ of vertices of weight $\geq k$. Further, $f_k = f_k(\Gamma)$ denotes the number of faces of weight $k$.

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Concerning the characterization of the weight vectors \((v_0, v_1, \ldots, v_{n-2})\) of arrangements of pseudocircles little is known. So far, sharp upper bounds on \(v_k\) exist only for \(k = 0\).

**Theorem 1.2** (Kedem et al. \cite{Kedem00}). For all arrangements of \(n \geq 3\) pseudocircles, 

\[
v_0 \leq 6n - 12.
\]

Moreover, for each \(n \geq 3\) there is an arrangement of \(n\) (proper) circles in the plane such that \(v_0 = 6n - 12\).

In a more common interpretation and originally motivated by motion planning problems \cite{Kedem00}, Theorem 1.2 shows that the complexity of the union of simple closed curves is linear in the number of elements, provided that each two curves have at most two points in common. At its core the inductive proof of Theorem 1.2 uses that by Euler’s formula planar graphs with \(n\) vertices have at most \(3n - 6\) edges. For arrangements with no vertices of weight \(> 0\), the intersection graph of the arrangement with vertex set \(V := \Gamma\) and edge set \(E := \{\{\gamma_i, \gamma_j\} \mid \gamma_i \cap \gamma_j \neq \emptyset\}\) can be shown to be planar. As each edge in \(E\) corresponds to two vertices in the arrangement, this gives the bound of \(6n - 12\). More complex arrangements with vertices of weight \(> 0\) can be disentangled so that the bound on \(v_0\) remains valid. By the circle packing theorem \cite{Fekete00}, any planar graph is the intersection graph of a circle packing and therefore (by slightly increasing the radii of the circles in the packing) also of an arrangement of (pseudo)circles. Consequently, Theorem 1.2 can be considered to be a generalization of Euler’s bound on the number of edges for planar graphs.

Theorem 1.2 is sharp for complete arrangements, too. That is, for each \(n \geq 3\) there is a complete arrangement with \(v_0 = 6n - 12\) as shown in Figure 1. The following general upper bound on \(v_{\leq k}\) can be obtained from Theorem 1.2 by some clever probabilistic methods.

**Theorem 1.3** (Sharir \cite{Sharer98}). For all arrangements of \(n\) pseudocircles and all \(k > 0\),

\[
v_{\leq k} \leq 26kn.
\]
We conjecture that the bound on $v_{\leq k}$ of Theorem 1.3 can be improved to $6(k + 1)n$. For $v_{\geq k}$, J. Linhart and Y. Yang established the following upper bound, which can be shown to be sharp for each $k$ with $0 \leq k \leq n - 2$, cf. [7].

**Theorem 1.4** (Linhart, Yang [7]). For all arrangements of $n \geq 2$ pseudocircles and all $k$ with $0 \leq k \leq n - 2$,

$$v_{\geq k} \leq (n + k)(n - k - 1).$$

## 1.1 Results

In this paper, we consider a question analogous to extremal graph theory: What is the maximal $v_0$ of a complete arrangement of $n$ pseudocircles not containing a subarrangement of a given type?

Evidently, arrangements of three pseudocircles are the smallest subarrangements of interest in this respect. Figure 2 shows the four different types one has to take into account.

![Types of subarrangements](image.png)

**Figure 2:** Complete arrangements of three pseudocircles in the plane.

Subarrangements of type $\alpha$ play a special role here, as they are the only arrangements of three pseudocircles which meet the bound of Theorem 1.2. The first result we are going to show is that $2(k + 1)n - k^2 - 3k - 1$ vertices of weight $k$ always force a subarrangement of type $\alpha$.

**Theorem 1.5.** Consider a complete arrangement of $n \geq 2$ pseudocircles that has no subarrangement of type $\alpha$, and let $0 \leq k \leq n - 2$. Then

$$v_{\leq k} \leq 2(k + 1)n - (k + 1)(k + 2).$$

Among all complete arrangements of four pseudocircles only one meets the bound of Theorem 1.2. In such an $\alpha^4$-arrangement each subarrangement of three pseudocircles is of type $\alpha$. Such arrangements prominently appear in the arrangement of Figure 1, where the three outer pseudocircles together with any other pseudocircle form an $\alpha^4$-arrangement. Our main result shows that $4n - 5$ vertices of weight 0 in a complete arrangement of $n$ pseudocircles force the existence of a subarrangement of type $\alpha^4$.

**Theorem 1.6.** In complete, $\alpha^4$-free arrangements of $n \geq 2$ pseudocircles,

$$v_0 \leq 4n - 6.$$
Theorems 1.5 and 1.6 correspond to well-known results of extremal graph theory. In particular, for $k = 0$ Theorem 1.5 gives a bound of $2n - 2$, which is analogous to the known bound of $n - 1$ on the number of edges in graphs with $n$ vertices not containing a topological $K_3$, the intersection graph of an $\alpha$-arrangement. Similarly, an $\alpha^4$-arrangement corresponds to $K_4$, or more precisely to a semitopological $S_3$. Thus, the bound in Theorem 1.6 is analogous to the bound of $2n - 3$ on the number of edges in graphs not containing a topological $K_4$ (cf. [2]), or a semitopological $S_3$ (cf. [10]). For further discussion see Section 5.

We note that the bounds of Theorems 1.5 and Theorem 1.6 are sharp. For Theorem 1.5 this follows from the sharpness of Theorem 1.4, which is used to prove Theorem 1.5, while for Theorem 1.6 the construction in Figure 3 gives complete, $\alpha^4$-free arrangements of $n \geq 2$ pseudocircles that meet the bound.

2 Forcing an $\alpha$-subarrangement (Proof of Theorem 1.5)

Arrangements of type $\alpha$ are the only complete arrangements of three pseudocircles without any face of weight 3, which is of importance in the light of the following Helly type theorem due to J. Molnár (see e.g. [1]).

**Theorem 2.1** (J. Molnár [8]). Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be an arrangement of pseudocircles such that for all pairwise distinct $\gamma_i, \gamma_j, \gamma_k$,

$$\text{int}(\gamma_i) \cap \text{int}(\gamma_j) \cap \text{int}(\gamma_k) \neq \emptyset.$$ 

Then $\bigcap_{i=1}^n \text{int}(\gamma_i)$ is nonempty and simply connected.

Thus, any complete, $\alpha$-free arrangement has a single face of weight $n$. For arrangements with this property the bounds of Theorems 1.2 and Theorems 1.3 can be improved. This improvement is based on Theorem 1.4 and the following result of Y. Yang.
Proposition 2.2 (Yang [11]). Let \( \Gamma \) be an arrangement of \( n \) pseudocircles with vertex weight vector \((v_0, v_1, \ldots, v_{n-2})\) and face weight vector \((f_0, f_1, \ldots, f_n)\) where \( f_n > 0 \). Then there is an arrangement of pseudocircles \( \Gamma' \) with vertex weight vector
\[
(v'_0, v'_1, \ldots, v'_{n-2}) = (v_{n-2}, v_{n-1}, \ldots, v_0)
\]
and face weight vector
\[
(f'_0, f'_1, \ldots, f'_n) = (f_n, f_{n-1}, \ldots, f_0).
\]

Proposition 2.2 can be shown by turning the arrangement in question inside out, so that any face of weight \( k \) becomes a face of weight \( n-k \). For a detailed proof see [6]. J. Linhart [5] pointed out that Proposition 2.2 together with Theorem 1.4 yields the following improvement of the upper bound on \( v \leq k \) for arrangements with \( f_n > 0 \).

Theorem 2.3 (Linhart [5]). For all arrangements of \( n \geq 2 \) pseudocircles with \( f_n > 0 \) and \( 0 \leq k \leq n-2 \),
\[
v_{\leq k} \leq 2(k+1)n - (k+1)(k+2).
\]

Proof. Let \( \Gamma \) be an arrangement with weight vector \((v_0, v_1, \ldots, v_{n-2})\) and \( f_n > 0 \). Then by Proposition 2.2, there exists an arrangement \( \Gamma' \) with \( v'_k := v_k(\Gamma') = v_{n-k-2} \) vertices of weight \( k \) for \( 0 \leq k \leq n-2 \). Therefore,
\[
v_{\leq k} = \sum_{j=0}^{k} v_j = \sum_{j=0}^{k} v'_{n-j-2} = \sum_{j=n-k-2}^{n-2} v'_j = v'_{\geq n-k-2}.
\]
Application of Theorem 1.4 then yields
\[
v_{\leq k} = v'_{\geq n-k-2} \leq (n + n - k - 2)(n - (n - k - 2) - 1)
= (2n - k - 2)(k + 1)
= 2(k + 1)n - (k + 1)(k + 2).
\]

Proof of Theorem 1.5. By Theorem 2.1, in any complete, \( \alpha \)-free arrangement, \( f_n > 0 \). Applying Theorem 2.3 yields the claimed bound.

3 Forcing an \( \alpha \)-subarrangement (Proof of Theorem 1.6)

In order to show Theorem 1.6, we start with the following result that will simplify matters, as it will be sufficient to derive a bound on \( f_0 \) to complete the proof.

Theorem 3.1. In any arrangement of \( n \geq 2 \) pseudocircles,
\[
v_0 \leq 2n + 2f_0 - 4.
\]

Theorem 3.1 will be proved without much effort from Theorem 1.2 with the aid of the subsequent upper bound on \( f_0 \) in Proposition 3.2, which is also an easy consequence of Theorem 1.2. As Theorem 3.1 together with Proposition 3.2 entails Theorem 1.2, this can be considered as self-strengthening of Theorem 1.2.
Proposition 3.2. For all arrangements of \( n \geq 3 \) pseudocircles,
\[
f_0 \leq 2n - 4.
\]

Proof. First note that the boundary of each bounded face of weight 0 consists of at least three edges (and hence vertices) of weight 0. For if there were a face with only two edges belonging to some pseudocircles \( \gamma_i \) and \( \gamma_j \), then \( \gamma_i \cap \gamma_j \) would have more than the two allowed intersection points. Concerning the unbounded face we will also assume that it has at least three edges on its boundary. In the case when it has only two edges belonging to some pseudocircles \( \gamma_i \) and \( \gamma_j \), then all other pseudocircles are contained in \( \text{int}(\gamma_i) \cup \text{int}(\gamma_j) \) so that \( f_0 = 1 \) and the bound trivially holds.

On the other hand, each vertex of weight 0 is on the boundary of only a single face of weight 0. Therefore by Theorem 1.2,
\[
f_0 \leq \frac{v_0}{3} \leq \frac{6n - 12}{3} = 2n - 4.
\]

In order to enhance readability, in the following we will use the terms 0-face and 0-edge to refer to faces and edges of weight 0, respectively. Further, \( \partial F \) shall denote the boundary of a face \( F \) in an arrangement.

Proof of Theorem 3.1. Without loss of generality, we only consider connected arrangements, that is, arrangements with connected arrangement graph. For arrangements with more than one component it is straightforward to modify them to make them connected by letting the components intersect in two additional vertices of weight 0, increasing \( v_0 \) but not changing \( f_0 \) and \( n \). The claimed bound then holds for the modified connected and hence also for the original arrangement.

We start showing the theorem for two particular cases. First, if the unbounded face has only two edges on its boundary, then as argued in the proof of Proposition 3.2 we have \( f_0 = 1 \) as well as \( v_0 = 2 \) and \( n \geq 2 \), so that the theorem holds.

Second, let us consider an arrangement \( \Gamma \) that has a 0-face \( F \) with more than three edges such that only three pseudocircles \( \gamma_i, \gamma_j, \gamma_k \) contribute edges to \( \partial F \). Then the same face \( F \) will appear in the subarrangement \( \Gamma' = \{ \gamma_i, \gamma_j, \gamma_k \} \). It is straightforward to verify that in all arrangements of three pseudocircles with a 0-face of more than three edges, \( f_0 = 1 \) and \( v_0 = 4 \). (Such arrangements are either complete arrangements of type \( \beta \), or arrangements with only four vertices all of which are of weight 0.) Thus, \( F \) is the only 0-face in \( \Gamma' \) and hence also in \( \Gamma \). Consequently, we have \( f_0 = 1 \), \( v_0 = 4 \), and \( n \geq 3 \) in \( \Gamma \), so that the theorem holds.

Thus, let us assume that \( \Gamma \) is an arrangement in which the unbounded 0-face and therefore all 0-faces have at least three edges on its boundary, and all 0-faces with more than three edges have more than three pseudocircles contributing to its boundary. Let \( f_{0,i} \) be the number of 0-faces with \( i \) edges. We prove the theorem by induction on the tuples \( (f_{0,n}, f_{0,n-1}, \ldots, f_{0,4}) \) in lexicographical order. First, if \( f_{0,i} = 0 \) for all \( i > 3 \), then all 0-faces are triangles, so that \( v_0 = 3f_0 \) and by Proposition 3.2,
\[
v_0 = 2f_0 + f_0 \leq 2f_0 + 2n - 4.
\]

\[\]
If there is at least one 0-face \( F \) with more than three edges, we may add a pseudocircle \( \gamma \) which intersects for each pseudocircle contributing to \( \partial F \) one boundary edge of \( F \) (cf. Figure 4, which shows the case when each pseudocircle contributes one edge to \( \partial F \)). By assumption, more than three pseudocircles contribute edges to \( F \), so that in the resulting arrangement \( \Gamma' \) there is an \( f_{0,i} (i > 3) \) being smaller than in \( \Gamma \), while only \( f_{0,j} \) with \( j < i \) may have increased. Therefore, the induction assumption is applicable. Now, assume that \( f_0 \) has increased by \( \ell \). Then \( v_0 \) has increased by \( 2(\ell + 1) \), so that applying the induction assumption to \( \Gamma' \) gives

\[
v_0(\Gamma) + 2(\ell + 1) = v_0(\Gamma') \leq 2(n + 1) + 2f_0(\Gamma') - 4 \\
= 2(n + 1) + 2(f_0(\Gamma) + \ell) - 4,
\]

whence the theorem follows.

Theorem 1.6 now is an immediate consequence of the following bound on \( f_0 \) (for a proof we refer to Section 4) together with Theorem 3.1.

**Theorem 3.3.** In complete, \( \alpha^4 \)-free arrangements of \( n \geq 2 \) pseudocircles,

\[
f_0 \leq n - 1.
\]

## 4 Proof of Theorem 3.3

### 4.1 Preliminaries

For \( \alpha \)-arrangements \( \Gamma_\alpha \) we denote the bounded 0-face of \( \Gamma_\alpha \) by \( \hat{F}_0(\Gamma_\alpha) \). Sloppily, we will often say that a point or face is *inside* \( \Gamma_\alpha \), meaning that it is contained in \( \hat{F}_0(\Gamma_\alpha) \). Correspondingly, by saying that something is *outside* \( \Gamma_\alpha \), we mean that it is contained in the unbounded 0-face of \( \Gamma_\alpha \). We start with some simple facts about the topology of (complete) arrangements. The aim is to establish that in each complete arrangement with more than one 0-face there is an \( \alpha \)-arrangement \( \Gamma_\alpha \), such that all bounded 0-faces are contained in \( \hat{F}_0(\Gamma_\alpha) \).
Observation 4.1. Let $\Gamma$ be an arrangement of pseudocircles and $\gamma \in \Gamma$. Then two 0-faces $F_1$, $F_2$ of $\Gamma$ are merged into one when removing $\gamma$ if and only if $\gamma$ has an edge $e_i$ of weight 0 on $\partial F_i$ ($i = 1, 2$) and $e_1$ and $e_2$ can be connected by a curve only contained in the interior of $\gamma$ (and especially not intersecting any other pseudocircle). In particular, such 0-faces $F_1$, $F_2$ exist when $f_0(\Gamma) > f_0(\Gamma \setminus \{\gamma\})$.

In the following, we will say that $\gamma$ separates $F$ or that $\gamma$ separates $F_1$ from $F_2$, if by removing $\gamma$ from $\Gamma$ two 0-faces $F_1$, $F_2$ are merged into a 0-face $F$ in $\Gamma \setminus \{\gamma\}$ with $F_1, F_2 \subseteq F$.

Proposition 4.2. Let $\Gamma$ be a complete arrangement of pseudocircles with $f_0(\Gamma) > 1$. Then $\Gamma$ has an $\alpha$-subarrangement.

Proof. We show that for $\alpha$-free arrangements $f_0 = 1$. Assume that there is an $\alpha$-free arrangement $\Gamma$ with $f_0(\Gamma) > 1$. By Theorem 2.1, $f_n(\Gamma) = 1$, so that we may apply Proposition 2.2 to conclude that there is an arrangement $\Gamma'$ with $f_n(\Gamma') > 1$. But this contradicts Theorem 2.1.

Lemma 4.3. Let $F^+$, $F^-$ be two 0-faces in some complete arrangement $\Gamma$ of pseudocircles. Then there is an $\alpha$-arrangement $\Gamma_\alpha \subseteq \Gamma$ such that $F^+$ is inside $\Gamma_\alpha$, while $F^-$ is outside $\Gamma_\alpha$ (or vice versa).

Proof. First remove pseudocircles from $\Gamma$ one after another, such that $F^+$ and $F^-$ are not merged. That is, when removing a pseudocircle we allow that $F^+$ or $F^-$ are merged with other 0-faces, but in the arising arrangement $F^+$ and $F^-$ must always be contained in different 0-faces. When it is no longer possible to remove a pseudocircle without merging $F^+$ and $F^-$, then in the resulting arrangement $\Gamma' = \{\gamma_1, \ldots, \gamma_k\}$ each remaining $\gamma_i$ separates $F^+$ from $F^-$. (Here and in the following $F^+$ and $F^-$ actually denote the possibly larger 0-faces in $\Gamma'$ containing the original faces $F^+$ and $F^-$. ) That is, on the boundary of each $\gamma_i$ there are two 0-edges $e_i^+$ and $e_i^-$ such that $e_i^+$ is on $\partial F^+$ and $e_i^-$ is on $\partial F^-$. By Proposition 4.2, $\Gamma'$ contains some $\alpha$-arrangement $\Gamma_\alpha = \{\gamma_{i_1}, \gamma_{i_2}, \gamma_{i_3}\}$. Assume that both $F^+$ and $F^-$ are contained in the same 0-face of $\Gamma_\alpha$. Then the edges $e_{i_1}^+, e_{i_1}^-, e_{i_2}^+, e_{i_2}^-, e_{i_3}^+, e_{i_3}^-$ are all contained on the same boundary curve $C$ of $\Gamma_\alpha$. However, it is not possible to distribute the edges $e_j^+, e_j^-$ on the pseudocircles $\gamma_j$ ($j = i_1, i_2, i_3$) such that the $e_j^+$ can be separated from the $e_j^-$ on $C$. It follows that $F^+$ and $F^-$ cannot be contained in the same 0-face of $\Gamma_\alpha$. 

Lemma 4.4. Let $F^+$, $F^-$ be two 0-faces in some complete arrangement $\Gamma$ that are both inside some $\alpha$-arrangement $\Gamma_\alpha \subseteq \Gamma$. Then there is another $\alpha$-arrangement $\Gamma_\alpha^+ \subseteq \Gamma$ with $F^+$ inside $\Gamma_\alpha^+$, and $F^-$ outside $\Gamma_\alpha^+$, such that all 0-faces of $\Gamma$ that are inside $\Gamma_\alpha^+$ are also inside $\Gamma_\alpha$.

Proof. By Lemma 4.3 there is an $\alpha$-arrangement $\Gamma'_\alpha$ such that $F^+$ is inside $\Gamma'_\alpha$ and $F^-$ is outside $\Gamma'_\alpha$ (or vice versa). Consider the arrangement $\Gamma'_\alpha \cup \Gamma_\alpha$ and note that $F^+$ and $F^-$ are contained in different 0-faces in $\Gamma'_\alpha \cup \Gamma_\alpha$. Remove pseudocircles from $\Gamma'_\alpha$ as long as the faces $F^+$, $F^-$ remain separated, and let $\Gamma'$ be a minimal subset of $\Gamma'_\alpha$ such that when removing a pseudocircle in $\Gamma'$ from $\Gamma'_\alpha \cup \Gamma_\alpha$ the faces $F^+$, $F^-$ are merged. (Again, $F^+$ and $F^-$ actually denote the possibly larger 0-faces in $\Gamma'_\alpha \cup \Gamma_\alpha$ containing the original faces $F^+$ and $F^-$ of $\Gamma$.)
If $\Gamma^*$ contains only a single element $\gamma$ then $F^+$ and $F^-$ are separated in $\Gamma_\alpha \cup \{\gamma\}$, and it is easy to see that $\gamma$ forms with suitable pseudocircles in $\Gamma_\alpha$ an $\alpha$-arrangement $\Gamma_\alpha^+$ with $F^+$ inside and $F^-$ outside. Furthermore, $\hat{F}_0(\Gamma_\alpha^+) \subseteq \hat{F}_0(\Gamma_\alpha)$ so that all 0-faces of $\Gamma$ that are inside $\Gamma_\alpha^+$ are also inside $\Gamma_\alpha$.

Thus, let us assume that $|\Gamma^*| \geq 2$. As no pseudocircle $\gamma \in \Gamma^*$ can be removed without merging $F^+$ and $F^-$, for each pseudocircle $\gamma_i \in \Gamma^*$ there are two 0-edges $e^+_i$, $e^-_i$ on the boundary of $\gamma_i$ such that $e^+_i$ is on $\partial F^+$, $e^-_i$ is on $\partial F^-$, and $e^-_i$ can be connected with $e^+_i$ by a curve only contained in $\text{int}(\gamma_i)$. Note that $e^+_i$ and $e^-_i$ must lie on the same component $C_i$ of $\gamma_i \cap \hat{F}_0(\Gamma_\alpha)$: Otherwise either $e^-_i$ cannot be connected with $e^+_i$ by a curve only contained in $\text{int}(\gamma_i)$, or $F^+$ and $F^-$ are separated in $\Gamma_\alpha \cup \{\gamma_i\}$, so that one could choose $\Gamma^* = \{\gamma_i\}$. Furthermore, $e^+_i$ and $e^-_i$ obviously cannot be neighbors on $C_i$, so that there must be a $\gamma_j \in \Gamma^*$ that separates $e^+_i$ from $e^-_i$ on $C_i$. Thus, by completeness of the arrangement, the situation is as shown in Figure 5(a) and $\Gamma^* = \{\gamma_i, \gamma_j\}$ by minimality of $\Gamma^*$. Note in particular that by Observation 4.1, $\gamma_j \cap \partial \hat{F}_0(\Gamma_\alpha)$ cannot lie in the interior of $\gamma_i$, as in this case it would be impossible to connect the edges $e^+_i$, $e^-_i$ by a curve only contained in the interior of $\gamma_i$. Similarly, $\gamma_i \cap \partial \hat{F}_0(\Gamma_\alpha)$ cannot lie in the interior of $\gamma_j$. Thus, for each $\gamma \in \Gamma_\alpha$ it holds that neither $\text{int}(\gamma)$ contains a point from $\gamma_i \cap \gamma_j$, nor can either $\text{int}(\gamma_i)$ or $\text{int}(\gamma_j)$ contain a point of $\gamma \cap \gamma_j$ or $\gamma \cap \gamma_i$, respectively. It follows that $\gamma_i$, $\gamma_j$ together with any $\gamma \in \Gamma_\alpha$ forms an $\alpha$-arrangement with either $F^+$ or $F^-$ inside (but not both). Note also that by Lemma 4.3 there is an $\alpha$-arrangement $\Gamma''_\alpha \subseteq \Gamma^* \cup \Gamma_\alpha$ with $F^+$ inside and $F^-$ outside. Hence, if $F^+$, $F^-$ are the only bounded 0-faces of $\Gamma^* \cup \Gamma_\alpha$ then $F^+$ is the only 0-face of $\Gamma^* \cup \Gamma_\alpha$ inside $\Gamma''_\alpha$ and we are done, since all 0-faces of $\Gamma$ that are inside $\Gamma''_\alpha$ are obviously also inside $\Gamma_\alpha$.

On the other hand, if there is another bounded 0-face $F \neq F^+, F^-$ in $\Gamma^* \cup \Gamma_\alpha$, this face can only be outside $\Gamma_\alpha$ with either $\gamma_i$ or $\gamma_j$ having at least one edge on its boundary. Let $\Gamma_\alpha = \{\gamma_1, \gamma_2, \gamma_3\}$ and assume without loss of generality that $\gamma_i \cap \partial F \neq \emptyset$. The only way $\gamma_i$ can have edges on bounded 0-faces inside and outside $\Gamma_\alpha$ is by forming a $\beta$-arrangement with two pseudocircles $\gamma'_{\ell}, \gamma''_{\ell} \in \Gamma_\alpha$ and intersecting the remaining $\gamma'_{m} \in \Gamma_\alpha$ in two vertices of weight 0. We deal with the exemplary case shown in Figure 5(b) with $\gamma'_{k} \cap \partial F^+ \neq \emptyset$. It is straightforward to verify that all other cases are symmetric. As argued above, $\gamma_j$ cannot intersect $\partial \hat{F}_0(\Gamma_\alpha)$ and hence in particular $\gamma'_{k}$ inside $\gamma_i$. Further, it is easy to see that $\gamma_j$ cannot intersect $\gamma'_{k}$ in two vertices on the outer boundary curve of $\Gamma_\alpha$ either. It follows that
\( \gamma_j \) intersects \( \gamma'_k \) either in at least one vertex of weight 0 on \( \partial F_0(\Gamma_\alpha) \) or in at least one point in \( \text{int}(\gamma'_m) \). In either case, \( F^+ \) is the only 0-face of \( \Gamma^* \cup \Gamma_\alpha \) inside \( \{\gamma_i, \gamma_j, \gamma'_k\} \) and all 0-faces of \( \Gamma \) that are inside \( \{\gamma_i, \gamma_j, \gamma'_k\} \) are also inside \( \Gamma_\alpha \).

Repeated application of Lemma 4.4 gives the following result.

**Corollary 4.5.** Let \( \Gamma \) be a complete arrangement with bounded 0-face \( F \). Then there is an \( \alpha \)-subarrangement \( \Gamma_\alpha \) in \( \Gamma \) such that \( F \) is the only 0-face inside \( \Gamma_\alpha \).

Projecting the arrangement to the sphere the unbounded face of weight 0 becomes bounded, while Corollary 4.5 is still applicable, which therefore yields the following counterpart to Corollary 4.5.

**Corollary 4.6.** Given a complete arrangement \( \Gamma \) with \( f_0(\Gamma) > 1 \), there is an \( \alpha \)-subarrangement \( \Gamma_\alpha \) in \( \Gamma \) such that all bounded 0-faces are inside \( \Gamma_\alpha \).

### 4.2 Peeling complete, \( \alpha^4 \)-free arrangements

Concluding, the following lemma allows to peel any complete, \( \alpha^4 \)-free arrangement by removing an outer pseudocircle together with at most one 0-face. Theorem 3.3 follows immediately.

**Lemma 4.7.** Let \( \Gamma \) be a complete, \( \alpha^4 \)-free arrangement with \( f_0(\Gamma) > 1 \), and let \( \Gamma_\alpha \) be an \( \alpha \)-subarrangement of \( \Gamma \) that contains all bounded 0-faces in its interior (cf. Corollary 4.6). Then there is a \( \gamma \in \Gamma_\alpha \) such that removing \( \gamma \) from \( \Gamma \) loses at most one face of weight 0, i.e.,

\[
f_0(\Gamma \setminus \{\gamma\}) \geq f_0(\Gamma) - 1.
\]

*Proof of Theorem 3.3.* The theorem is trivial for \( n = 2 \), while for \( n = 3 \) it can be verified from Figure 2. We proceed by induction on \( n \). If \( f_0(\Gamma) \leq 1 \) the theorem again holds trivially for \( n \geq 2 \). If \( f_0(\Gamma) > 1 \) then Lemma 4.7 allows to remove a pseudocircle \( \gamma \), so that \( f_0 \) is decreased by at most 1. Together with the induction assumption for \( \Gamma \setminus \{\gamma\} \) this shows the theorem.

*Proof of Lemma 4.7.* We derive a contradiction from the assumption that for each \( \gamma_i \in \Gamma_\alpha = \{\gamma_1, \gamma_2, \gamma_3\} \) it holds that \( f_0(\Gamma \setminus \{\gamma_i\}) < f_0(\Gamma) - 1 \) (i.e., removing any pseudocircle in \( \Gamma_\alpha \) from \( \Gamma \) decreases \( f_0 \) by at least 2). Under this assumption, each \( \gamma_i \in \Gamma_\alpha \) borders on at least three 0-faces, two of which (by assumption on \( \Gamma_\alpha \)) are contained in \( F_0(\Gamma_\alpha) \). According to Observation 4.1, these two faces \( F_{i,1}, F_{i,2} \) can be connected by a curve \( C_i \) only contained in the interior of \( \gamma_i \). Each \( C_i \) separates the interior of \( \gamma_i \) into two regions \( R_i^-, R_i^+ \), where we assume \( R_i^+ \) to be the region that borders on the unbounded face of weight 0, see Figure 6(a).

On the other hand, \( F_{i,1}, F_{i,2} \) must be separated by some other pseudocircle: For each pseudocircle \( \gamma_i \) in \( \Gamma_\alpha \) there is a pseudocircle \( \gamma'_i \notin \Gamma_\alpha \) that intersects \( \gamma_i \) in \( R_i^- \) \( (i = 1, 2, 3) \). Any such \( \gamma'_i \) separates \( F_{i,1} \) from \( F_{i,2} \). Note that both \( \{\gamma_i, \gamma'_i, \gamma_j\} \) \( (j \neq i) \) and \( \{\gamma_i, \gamma'_i, \gamma_k\} \) \( (k \neq i, j) \) are \( \alpha \)-arrangements, since \( \gamma'_i \) cannot intersect \( \gamma_j \) in the interior of \( \gamma_i \) without crossing \( C_i \) (which cannot happen according to Observation 4.1). Summarizing, one of the two 0-faces \( F_{i,1}, F_{i,2} \) is contained inside the \( \alpha \)-arrangement \( \{\gamma_i, \gamma'_i, \gamma_j\} \), the other contained inside the \( \alpha \)-arrangement \( \{\gamma_i, \gamma'_i, \gamma_k\} \) (see Figure 6(a)).
We distinguish the following two cases:

\( |\{\gamma_1', \gamma_2', \gamma_3'\}| = 1: \)

In this case \( \gamma_1' = \gamma_2' = \gamma_3' \), so that \( \Gamma_\alpha \cup \{\gamma_1'\} \) is an \( \alpha^4 \)-arrangement, contradicting our assumption.

\( |\{\gamma_1', \gamma_2', \gamma_3'\}| \geq 2: \)

Assume without loss of generality that \( \gamma_1' \neq \gamma_2' \). Let us consider the intersections of \( \gamma_j' \) with \( \gamma_i \) for \( i, j \in \{1, 2\} \) and \( i \neq j \). First note that int(\( \gamma_j' \)) cannot intersect both \( R_i^+ \) and \( R_i^- \), as this would cause a forbidden intersection with \( C_i \). Now, if \( \gamma_j' \) intersects \( R_i^- \), \( \gamma_j' \) cannot intersect \( \gamma_3 \) in the interior of either \( \gamma_1 \) or \( \gamma_2 \) without intersecting either \( C_1 \) or \( C_2 \). Therefore, by completeness \( \{\gamma_1, \gamma_2, \gamma_3, \gamma_j'\} \) is a forbidden \( \alpha^4 \)-arrangement, cf. Figure 6(b).

Thus, let us assume that \( \gamma_j' \) intersects \( R_i^+ \) and \( \gamma_j' \) intersects \( R_i^+ \), and consequently \( \text{int}(\gamma_j') \cap R_i^- = \emptyset \) as well as \( \text{int}(\gamma_j') \cap R_i^+ = \emptyset \). Since we also have \( \text{int}(\gamma_j') \cap R_i^+ = \emptyset \) for \( i = 1, 2 \), it follows that \( \text{int}(\gamma_1') \cap \text{int}(\gamma_2') \cap \text{int}(\gamma_1) = \emptyset \) and \( \text{int}(\gamma_1') \cap \text{int}(\gamma_2') \cap \text{int}(\gamma_2) = \emptyset \). Therefore \( \{\gamma_1', \gamma_2', \gamma_1\} \) and \( \{\gamma_1', \gamma_2', \gamma_2\} \) are \( \alpha \)-arrangements. We have already argued above that \( \{\gamma_1, \gamma_1', \gamma_2\} \) and \( \{\gamma_2, \gamma_2', \gamma_1\} \) are \( \alpha \)-arrangements as well, so that \( \{\gamma_1, \gamma_2, \gamma_1', \gamma_2'\} \) is a forbidden \( \alpha^4 \)-arrangement.

5 Discussion and Open Problems

5.1 Implications

The improved upper bound on \( v_0 \) of Theorem 1.6 can in turn be used to improve the upper bound on \( v_{\leq k} \) for complete, \( \alpha^4 \)-free arrangements.

Theorem 5.1. For complete, \( \alpha^4 \)-free arrangements of \( n \geq 2 \) pseudocircles and all \( k > 0 \),

\( v_{\leq k} \leq 18kn \).

Figure 6: Illustration of the situation in Lemma 4.7. The curves \( C_i \) separate \( \text{int}(\gamma_i) \) into regions \( R_i^+ \) and \( R_i^- \) (dotted). The pseudocircles \( \gamma_j' \) intersect \( \gamma_i \) in \( R_i^- \). If \( \gamma_j' \) intersects \( R_i^- \) as in the right picture, \( \gamma_3 \) can be intersected only by allowing an \( \alpha^4 \)-arrangement.
Proof. The proof is basically identical to the proof of Theorem 1.3 in [9], only with the application of Theorem 1.2 replaced by an application of Theorem 1.6 and the constants adapted accordingly.

Remark 5.2. As \( \alpha^4 \)-arrangements cannot be realized with unit circles, Theorems 1.6 and 5.1 hold in particular for complete arrangements of unit circles. Note that for general arrangements of unit circles no significant improvement of Theorem 1.2 is possible: Consider the infinite arrangement obtained from a densest circle packing by increasing the radius of the circles by a sufficiently small value, so that each touching point is transformed into two intersection points of weight 0. Then each circle intersects six others, and choosing a suitable finite subarrangement of \( n \) pseudocircles gives a total of \( \approx 6n \) vertices of weight 0.

5.2 Generalization

A natural question is whether the results of Theorems 1.5 and 1.6 can be generalized to arbitrary (i.e., not necessarily complete) arrangements. We conjecture that a pendant to Theorem 1.5 holds for arrangements that do not contain a cyclic chain according to the following definition.

Definition 5.3. A cyclic chain is an arrangement \( \Gamma \) whose elements can be indexed such that
\[
\Gamma = \{ \gamma_1, \ldots, \gamma_n \},
\]
and \( \gamma_i \) intersects \( \gamma_{i+1} \) in two vertices of weight 0 for all \( i = 1, \ldots, n \) (with \( \gamma_{n+1} := \gamma_1 \)). Also, there are no further intersections.

Our current proof of Theorem 1.5 cannot be easily adapted to show this conjecture. Still, if one is able to generalize the result of Theorem 1.4 to certain arrangements on the sphere (it is easy to see that Theorem 1.4 cannot hold for all arrangements on the sphere), a modification may succeed.

Concerning Theorem 1.6, it is not even clear what a generalization may look like. An obvious generalization of an \( \alpha^4 \)-arrangement would be that of a wheel.

Definition 5.4. A wheel is an arrangement which consists of a cyclic chain \( \Gamma' \) plus another pseudocircle \( \gamma \) which separates one of the two faces of weight 0 in \( \Gamma' \) into at least three faces of weight 0.

Indeed, for arrangements in which all vertices have weight 0, the following result can be considered as a generalization of Theorem 1.6. It follows immediately by application of the following, already mentioned result of C. Thomassen [10] to the arrangement’s intersection graph (as defined after Theorem 1.2): Any graph \( G = (V, E) \) that does not contain a semitopological graph \( S_3 \) (i.e., a cycle plus an additional vertex that is connected to three vertices in the cycle) has at most \( 2|V| - 3 \) edges.

Proposition 5.5. For wheel-free arrangements of \( n \geq 2 \) pseudocircles with no vertices of weight \( > 0 \),
\[
v_0 \leq 4n - 6.
\]
Figure 7: Difficulties when trying to generalize Theorem 1.6.

However, Proposition 5.5 does not hold in general, as the arrangement in Figure 7(a) shows. Although the arrangement does not contain a wheel, the bounds $f_0 \leq n-1$ and $v_0 \leq 4n - 6$ holds. On the other hand, simply allowing further intersection points in the definition of cyclic chain is too generous. While it is possible to obtain a variant of Theorem 1.6 for arbitrary arrangements, this result is not a proper generalization of Theorem 1.6: The (complete) arrangement in Figure 7(b) is $\alpha^4$-free, but not wheel-free according to the modified definition. Thus, it is not clear what a good generalization of Theorem 1.6 may look like.

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References


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As the two circles intersect (inside the ellipse), the outer pseudocircles do not form a cyclic chain as given by Definition 5.3.


