A FIXED-PARAMETER ALGORITHM FOR THE MINIMUM MANHATTAN NETWORK PROBLEM

Christian Knauer* and Andreas Spillner†

ABSTRACT. A Manhattan network for a finite set $P$ of $n$ points in the plane is a geometric graph such that its vertex set contains $P$, its edges are axis-parallel and non-crossing and, for any two points $p$ and $q$ in $P$, there exists a path in the network connecting $p$ and $q$ whose length equals the $l_1$-distance between $p$ and $q$. The problem of computing a Manhattan network of minimum total edge length for a given point set $P$ has recently been shown to be NP-hard. In this paper, using as the parameter the minimum number $h$ of horizontal straight lines that contain the points in $P$, we present a fixed-parameter algorithm for this problem running in $O^*(2^{14h})$ time (neglecting a factor that is polynomial in $n$) and note that, under the exponential time hypothesis for 3-SAT, a run time that is subexponential in $h$ is impossible.

1 Introduction

A Manhattan network is a finite graph $N = (V, E)$ whose vertex set $V \subseteq \mathbb{R}^2$ consists of a set of points in the plane such that, for every edge $e = \{p, q\}$ in $E$, the straight line segment $\overline{p, q}$ with endpoints $p$ and $q$ is either horizontal or vertical and, for any two distinct edges $e_1 = \{p_1, q_1\}$ and $e_2 = \{p_2, q_2\}$ in $E$, the straight line segments $\overline{p_1, q_1}$ and $\overline{p_2, q_2}$ do not cross, that is, $\overline{p_1, q_1} \cap \overline{p_2, q_2} \subseteq e_1 \cap e_2$ holds. Defining the length $\ell(e)$ of an edge $e = \{p, q\}$ in $E$ to be the $l_1$-distance $l_1(p, q)$ between the points $p$ and $q$ and, as usual, the length $\ell(p)$ of a path $p = p_0, p_1, \ldots, p_k$ in $N$ as the sum $\sum_{i=1}^{k} \ell(\{p_{i-1}, p_i\})$ of the lengths of the edges on $p$, we call $p$ a monotone path in $N$ if $l_1(p_0, p_k) = \ell(p)$ holds. In addition, given a finite set of points $P \subseteq \mathbb{R}^2$, we define a Manhattan network for $P$ as a Manhattan network $N = (V, E)$ with $P \subseteq V$ such that for any two distinct $p, q \in P$ there exists a monotone path from $p$ to $q$ in $N$ (an example of such a network is depicted in Figure 1(a)). Finally, such a network is called minimum if its total edge length $\lambda(N)$ is minimum among all Manhattan networks for $P$.

The problem of computing a minimum Manhattan network for a given point set $P$, known in the literature as the minimum Manhattan network problem (MMN), has only recently been shown to be NP-hard and also the existence of an FPTAS for MMN would imply P=NP [8]. Most previous work focused on approximation algorithms for MMN [4, 5, 7, 13, 14, 15, 16, 19]. All these algorithms yield a constant-factor approximation. The best approximation factor so far, namely 2, is guaranteed for an algorithm based on rounding the solution of a suitable linear program [7], for an algorithm based on applying the...
primal-dual method to the same linear program [18], and for a greedy algorithm [14]. The latter two algorithms run in $O(n \log n)$ time for a set $P$ containing $n$ points. Other previous work on MMN includes the generalization of the problem to 3-dimensional space [17] and in [7] a variant of MMN was suggested where only a certain subset of the pairs of points in $P$ must be connected by a monotone path in the Manhattan network. A generalization of this variant to 3-dimensional space is studied in [10].

Concerning exact algorithms for MMN, an approach based on a mixed-integer linear program formulation is presented in [5], where it is also posed as an open problem to design a fixed-parameter algorithm for MMN. In this paper we will present such an algorithm using as the parameter the minimum number $h = h(P)$ of horizontal straight lines such that every point in $P$ is contained in one of these straight lines. This parameter has previously been used to design fixed-parameter algorithms for problems similar to MMN such as, for example, the Minimum Rectilinear Steiner Tree problem (MRST). For MRST, an algorithm running in $O^*(16^h)$ time is outlined in [1] and in [6] the run time has been improved to $O^*(10^h)$ (the $O^*$-notation neglects polynomial factors). Interestingly, the parameter $h$ has also been considered in the context of parameterizing problems that admit a polynomial time algorithm but are hard to parallelize (see e.g. [12]). The parameter $h$ is also related to the concept of $r$-outerplanar graphs introduced in [3], which proved useful in the design of fixed-parameter algorithms on planar graphs (see e.g. [2]), and to the number of layers used as a parameter in layered graph drawing problems (see e.g. [9]).

The paper is structured as follows. In Section 2, after introducing some more notation, we present a fixed-parameter algorithm for MMN. While, in view of the existing fixed-parameter algorithms for MRST, it is probably not surprising that MMN is also fixed-parameter tractable with respect to the parameter $h$, it is not immediately clear that a run time in $O^*(c^h)$ for some constant $c$ can be achieved. This is the main result of the paper and will be established in Section 3. We conclude in the last section, briefly observing that a run time that is subexponential in $h$ is impossible under the exponential time hypothesis for 3-SAT. The reader not familiar with this hypothesis or parameterized complexity theory is referred to [11].

2 Description of the algorithm

As above, let $P$ denote the given set of points in the plane. For any point $q \in \mathbb{R}^2$ we denote by $x(q)$ and $y(q)$ the $x$- and $y$-coordinate of $q$, respectively. Let $x_1, \ldots, x_l$ denote the increasingly sorted sequence of $x$-coordinates in $X := \{x(p) : p \in P\}$ and define, for any $i \in \{1, \ldots, l\}$, the set $P_i = \{p \in P : x(p) \leq x_i\}$. Similarly, let $y_1, \ldots, y_h$ denote the increasingly sorted sequence of $y$-coordinates in $Y := \{y(p) : p \in P\}$. Note that, by definition, the $y$-coordinates in $Y$ correspond to the $h$ horizontal straight lines $L_1, \ldots, L_h$, $L_j := \{q \in \mathbb{R}^2 : y(q) = y_j\}$, $j \in \{1, \ldots, h\}$, that contain the point set $P$. In the following we will use the fact that there always exists a minimum Manhattan network $\mathcal{N} = (V, E)$ for $P$ with $V \subseteq X \times Y$ (see e.g. [7, Lemma 2.1]), that is, $\mathcal{N}$ is a subgraph of the grid induced by the points in $P$ (cf. Figure 1(b)).

Our algorithm sweeps over the point set $P$ from left to right. For every $i \in \{1, \ldots, l\}$
we compute a collection \( \mathcal{C}_i \) of Manhattan networks for \( P_i \). Define the points \( v_{i,j} := (x_i, y_j) \), \( j \in \{1, \ldots, h\} \), and put \( V_i := \{v_{i,j} : 1 \leq j \leq h\} \). For each Manhattan network \( \mathcal{N} \) in \( \mathcal{C}_i \) we keep track of those points \( v \) in \( V_i \) that are vertices of \( \mathcal{N} \) and, in case they are, for which points \( q \) in \( P_i \) there exists a monotone path from \( v \) to \( q \) in \( \mathcal{N} \). This information will be used in the computation of the collection \( \mathcal{C}_{i+1} \) from the collection \( \mathcal{C}_i \). For example, in Figure 1(c) a Manhattan network \( \mathcal{N} \) for \( P_6 \) is depicted that contains, in addition to the point \( v_{6,1} \) that must be a vertex of \( \mathcal{N} \) by definition, the points \( v_{6,3} \) and \( v_{6,4} \) as vertices but not the point \( v_{6,2} \). The points \( q \) in \( P_6 \) for which there exists a monotone path in \( \mathcal{N} \) from \( v_{6,4} \) to \( q \) are precisely those with \( x(q) \leq x_5 \) and \( y(q) \leq y_3 \).

To describe the collection \( \mathcal{C}_i \) more formally, define, for all \( i \in \{1, \ldots, l\} \) and all \( j \in \{1, \ldots, h\} \), the set \( R_{i,j} \) containing the rightmost point in \( P_i \) that lies on line \( L_j \), denoted by \( r_{i,j} \), in case such a point exists. Otherwise the set \( R_{i,j} \) is empty. So, the set \( R_{i,j} \) is either empty or contains a single point. If it is not empty the point \( r_{i,j} \) represents all points \( q \) in \( P_i \) with \( x(q) \leq x(r_{i,j}) \) and \( y(q) = y(r_{i,j}) \) in the following sense: For any point \( q' \in \mathbb{R}^2 \) with \( x(q') \geq x(r_{i,j}) \), a Manhattan network for \( P_i \) that contains a monotone path from \( q' \) to \( r_{i,j} \) also contains a monotone path from \( q' \) to any point \( q \) represented by \( r_{i,j} \). In particular, this implies the following fact that will be used in our algorithm.

**Observation 1.** When describing for a Manhattan network \( \mathcal{N} \) in \( \mathcal{C}_i \) the points \( q \) in \( P_i \) for which there exists a monotone path from \( v_{i,j} \) to \( q \) in \( \mathcal{N} \), \( j \in \{1, \ldots, h\} \), it suffices to keep track of those points \( q \) in \( \cup_{k=1}^h R_{i,k} \) for which such a path exists.

Next, motivated by Observation 1, we introduce some notation to capture for which points \( q \) in \( \cup_{k=1}^h R_{i,k} \) there exists a monotone path from \( v_{i,j} \), \( j \in \{1, \ldots, h\} \), to \( q \) in a Manhattan network \( \mathcal{N} \) for \( P_i \). It will be convenient to further distinguish, among the points \( q \) in \( \cup_{k=1}^h R_{i,k} \) for which such a path exists, between those with \( y(q) \geq y(v_{i,j}) \) and those with \( y(q) < y(v_{i,j}) \). To this end, we put, for every \( i \in \{1, \ldots, l\} \), \( R_i := \{j \in \{1, \ldots, h\} : R_{i,j} \neq \emptyset\} \) and define a pair \( \Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h)) \) in which, for all \( j \in \{1, \ldots, h\} \), \( A_j \) is a subset of \( \{j, j+1, \ldots, h\} \cap R_i \) and \( B_j \) is a subset of \( \{1, 2, \ldots, j-
\textbf{COMPUTE-MMN}(P)

\begin{itemize}
\item \textbf{Input:} a set $P \subseteq \mathbb{R}^2$ of $n$ points
\item \textbf{Output:} a minimum Manhattan network for $P$
\end{itemize}

\begin{enumerate}
\item Compute $X = \{x_1, \ldots, x_l\}$ and $Y = \{y_1, \ldots, y_h\}$.
\item Initialize an empty collection $C_1$.
\item \textbf{for each} pair $\Pi$ that is admissible for $P_1$ \textbf{do}
\item \hspace{1em} Compute a minimum Manhattan network $N$ for $P_1$ and $\Pi$.
\item \hspace{1em} Add $N$ to $C_1$.
\item \textbf{for} $i = 1$ to $l - 1$ \textbf{do}
\item \hspace{1em} Initialize an empty collection $C_{i+1}$.
\item \textbf{for each} $\mathcal{N}$ in $C_i$ \textbf{do}
\item \hspace{1em} \textbf{for each} $H \subseteq E_{i+1}$ \textbf{do}
\item \hspace{2em} Form the Manhattan network $\mathcal{N}'$ by adding the edges in $H$ to $\mathcal{N}$.
\item \hspace{2em} \textbf{if} $\mathcal{N}'$ is a Manhattan network for $P_{i+1}$ \textbf{then}
\item \hspace{3em} \textbf{if} $C_{i+1}$ contains a Manhattan network $\mathcal{N}''$ \textbf{with} $\pi(\mathcal{N}') = \pi(\mathcal{N}'')$ \textbf{then}
\item \hspace{4em} \textbf{if} $\lambda(\mathcal{N}'') > \lambda(\mathcal{N}')$ \textbf{then}
\item \hspace{5em} Remove $\mathcal{N}''$ from $C_{i+1}$ and add $\mathcal{N}'$ to $C_{i+1}$.
\item \hspace{4em} \textbf{else}
\item \hspace{5em} Add $\mathcal{N}'$ to $C_{i+1}$.
\item \hspace{1em} \textbf{return} a Manhattan network $\mathcal{N}$ in $C_i$ with $\lambda(\mathcal{N})$ minimum.
\end{enumerate}

Figure 2: Pseudocode for our algorithm for computing a minimum Manhattan network.

$1 \} \cap R_i$ to be \textit{admissible} for $P_i$. The collection $C_i$ contains, for a set of certain pairs $\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))$ that are admissible for $P_i$ and that we will describe below, a Manhattan network that has minimum total edge length among those Manhattan networks for $P_i$ that contain, for all $j \in \{1, \ldots, h\}$ and all $k \in A_j \cup B_j$, a monotone path from $v_{i,j}$ to $r_{i,k}$. We call such a network a \textit{minimum Manhattan network} for $P_i$ and $\Pi$. Note that, conversely, one can associate with every Manhattan network $\mathcal{N}$ for $P_i$ a \textit{canonical} pair $\pi(\mathcal{N}) = ((A_1(\mathcal{N}), \ldots, A_h(\mathcal{N})), (B_1(\mathcal{N}), \ldots, B_h(\mathcal{N})))$ that is admissible for $P_i$ by putting $A_j(\mathcal{N})$ to be the set of those $k \in R_i$ for which there exists a monotone path in $\mathcal{N}$ from $v_{i,j}$ to $r_{i,k}$ and $y(r_{i,k}) \geq y(v_{i,j})$ holds, and, similarly, putting $B_j(\mathcal{N})$ to be the set of those $k \in R_i$ for which there exists a monotone path in $\mathcal{N}$ from $v_{i,j}$ to $r_{i,k}$ and $y(r_{i,k}) < y(v_{i,j})$ holds, $j \in \{1, \ldots, h\}$. For example, with the Manhattan network for $P_0$ depicted in Figure 1(c), is associated the canonical admissible pair $\pi((\{1, 2, 3, 4\}, \emptyset, \{3, 4\}, \emptyset, \emptyset, \{2\}, \{2, 3\}))$.

Our algorithm for computing a minimum Manhattan network for a given point set $P$ is summarized in Figure 2. After computing the sets of coordinates $X$ and $Y$ (line 1), which provides the information needed to perform the sweep, the collection $C_1$ is computed (lines 2-5). For $C_1$ we consider every pair $\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))$ that is admissible for $P_1$. To compute a minimum Manhattan network for $P_1$ and $\Pi$ we first construct the set $U(\Pi) := \{v_{1,j} \in V_1 : A_j \cup B_j \neq \emptyset\}$, that is, the set of those points in $V_1$ that must be
Figure 3: (a) The edges of the grid induced by $P$ that are contained in $E_7$ are drawn bold. (b) The Manhattan network obtained by adding a specific subset $H \subseteq E_7$ to the Manhattan network in Figure 1(c). In this case, the resulting network is not a Manhattan network for $P_7$.

connected to some point in $P_1$ in the Manhattan network. Then we compute the points $v_{\min}$ and $v_{\max}$ in $P_1 \cup U(\Pi)$ with minimum and maximum $y$-coordinate, respectively. It is not hard to see that the minimum Manhattan network $N = (V, E)$ for $P_1$ and $\Pi$ is unique with $V := \{v \in V_1 : y(v)_{\min} \leq y(v) \leq y(v)_{\max}\}$ and $E$ containing the edges of the grid induced by $P$ connecting the vertices in $V$.

The actual sweep is described in lines 6-16. The basic idea is to consider each Manhattan network in the collection $C_i$ and extend it by adding edges from the set (cf. Figure 3(a))

$$E_{i+1} := \{\{v_{i,j}, v_{i+1,j}\} : j \in \{1, \ldots, h\}\} \cup \{\{v_{i+1,j}, v_{i+1,j+1}\} : j \in \{1, \ldots, h-1\}\}.$$ 

So, for each $N$ in $C_i$ and each subset $H \subseteq E_{i+1}$ we obtain a Manhattan network $N'$ that is a subgraph of the grid induced by the points in $P_{i+1}$ (cf. Figure 3(b)). Among the Manhattan networks $N'$ obtained by this construction, we are only interested in those that are a Manhattan network for $P_{i+1}$. For these we distinguish two cases. If the collection $C_{i+1}$ contains already a Manhattan network $N''$ for $P_{i+1}$ whose associated canonical pair $\pi(N'')$ equals $\pi(N')$ and, in addition, the total edge length $\lambda(N'')$ of $N''$ is smaller than the total edge length $\lambda(N')$ of $N'$, then we replace $N''$ by $N'$ in $C_{i+1}$. Otherwise we simply add $N'$ to $C_{i+1}$. As can be seen, the set of admissible pairs $\Pi$ for $P_{i+1}$ for which $C_{i+1}$ contains a Manhattan network $N$ with $\Pi = \pi(N)$ is implicitly determined by our algorithm. When the sweep has finished, we return a Manhattan network of minimum total length in $C_l$ (line 17). This concludes the description of our algorithm.

3 Analysis of the algorithm

Before we start the analysis, we introduce some more notation that is motivated by the fact that, as we will see below, not all admissible pairs are actually relevant. To make this more precise, let $\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))$ and $\Pi' = ((A'_1, \ldots, A'_h), (B'_1, \ldots, B'_h))$ be two admissible pairs for $P_i$, $i \in \{1, \ldots, l\}$. We say that $\Pi$ makes $\Pi'$ redundant if $A'_j \subseteq A_j$ and
\(B'_j \subseteq B_j\) holds for all \(j \in \{1, \ldots, h\}\) and every minimum Manhattan network for \(P_i\) and \(\Pi'\) is also a minimum Manhattan network for \(P_i\) and \(\Pi\). Intuitively, viewing \(\Pi'\) as being formed by removing elements from some of the sets \(A_j\) or \(B_j\) in \(\Pi\), the total edge length of a minimum Manhattan network for \(P_i\) and \(\Pi'\) might be strictly smaller than the total edge length of a minimum Manhattan network for \(P_i\) and \(\Pi\). However, if it stays the same then considering \(\Pi'\) is not really necessary because with \(\Pi\) even more connections via monotone paths from points in \(V_i\) to points in \(P_i\) are guaranteed without increasing the total edge length.

The following lemma forms the basis of the correctness proof for our algorithm. In it we refer to the concept of a set \(A\) of admissible pairs for \(P_i\), \(i \in \{1, \ldots, l\}\), to be a cover, that is, for every pair \(\Pi'\) that is admissible for \(P_i\) there exists some pair \(\Pi\) in \(A\) that makes \(\Pi'\) redundant.

Lemma 1. Let \(i \in \{1, 2, \ldots, l - 1\}\). Assume that \(C_i\) is such that \(A_i := \{\pi(N) : N \in C_i\}\) is a cover and for every \(\Pi \in A_i\) there is a minimum Manhattan network for \(P_i\) and \(\Pi\) in \(C_i\). Then there exists, for every pair \(\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))\) that is admissible for \(P_{i+1}\), a Manhattan network \(N\) in \(C_i\) and a subset \(H \subseteq E_{i+1}\) such that the Manhattan network \(\tilde{N}\) formed by adding the edges in \(H\) to \(N\) is a minimum Manhattan network for \(P_{i+1}\) and \(\Pi\).

Proof: Consider an arbitrary minimum Manhattan network \(\tilde{N} = (\tilde{V}, \tilde{E})\) for \(P_{i+1}\) and \(\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))\). Let \(N^* = (V^*, E^*)\) denote the subnetwork of \(\tilde{N}\) induced by those vertices \(v \in \tilde{V}\) with \(x(v) \leq x_i\). Since \(\tilde{N}\) is a Manhattan network for \(P_{i+1}\), the network \(N^*\) is a Manhattan network for \(P_i\).

Now, consider the canonical pair \(\Pi^* = ((A_1^*, \ldots, A_h^*), (B_1^*, \ldots, B_h^*)) := \pi(N^*)\) associated to \(N^*\) that is admissible for \(P_i\). Since, by assumption, \(A_i\) is a cover, \(\Pi^*\) is made redundant by some \(\Pi \in A_i\). Moreover, also by assumption, the collection \(C_i\) contains a minimum Manhattan network \(N\) for \(P_i\) and \(\Pi\) and, thus, \(\lambda(N^*) \geq \lambda(N)\) must hold.

Next recall that our algorithm will consider the network \(N''\) formed by adding to \(N\) the edges in the set \(H := E_{i+1} \cap \tilde{E}\) consisting of those edges in \(E_{i+1}\) that are edges of the network \(\tilde{N}\). We claim that \(N''\) is a minimum Manhattan network for \(P_{i+1}\) and \(\Pi\).

First we show that \(N''\) is a Manhattan network for \(P_{i+1}\): Since \(N\) is a Manhattan network for the points in \(P_i\) and both \(\tilde{N}\) and \(N''\) contain the same subset \(H\) of edges in \(E_{i+1}\), the only possibility for \(N''\) to fail to be a Manhattan network for \(P_{i+1}\) is that there exists some \(j \in \{1, \ldots, h\}\) and a point \(q \in P_i\) such that \(v_{i+1,j}\) is a point in \(P_{i+1}\) and there is no monotone path from \(v_{i+1,j}\) to \(q\) in \(N''\). To see that this situation cannot occur, consider an arbitrary \(q \in P_i\) and an arbitrary \(v \in V_{i+1} \cap P_{i+1}\). Let \(k\) be such that \(y(q) = y_k\). Since \(\tilde{N}\) is a Manhattan network for \(P_{i+1}\), it contains a monotone path \(p\) from \(v\) to \(r_{i,k}\). Let \(u\) denote the first vertex on \(p\) that we meet when we walk along \(p\) from \(v\) to \(r_{i,k}\) which belongs to \(V_i\). Then the network \(N^*\) contains a monotone path from \(u\) to \(r_{i,k}\). Thus, by the definition of the canonical pair \(\Pi^*\) and the fact that \(\Pi^*\) is made redundant by \(\Pi\), it follows that the network \(N\) contains a monotone path from \(u\) to \(r_{i,k}\). Furthermore, since \(\tilde{N}\) is a Manhattan network for \(P_i\), there exists a monotone path from \(r_{i,k}\) to \(q\) in \(\tilde{N}\). Thus, \(N''\) contains a monotone path from \(u\) to \(q\) and, therefore, \(N''\) contains a monotone path from \(v\) to \(q\). Hence \(N''\) is indeed a Manhattan network for \(P_{i+1}\).
A similar argument establishes that the network $\mathcal{N}'$ satisfies the additional requirements imposed by the admissible pair $\tilde{\Pi}$. Consider any $j \in \{1, \ldots, h\}$ and any $k \in A_j \cup B_j$. The network $\tilde{\mathcal{N}}$ contains a monotone path $p$ from $v_{i+1,j}$ to $r_{i+1,k}$. If this path $p$ uses only edges in $E_{i+1}$ it is, by construction, also contained in $\mathcal{N}'$. Otherwise let $u$ denote the first vertex on $p$ that we meet when we walk along $p$ from $v_{i+1,j}$ to $r_{i+1,k}$ which belongs to $V_i$. There is a monotone path from $u$ to $r_{i+1,k}$ in $\mathcal{N}'$ and, therefore, by construction, also in $\mathcal{N}$. This yields a monotone path from $v_{i+1,j}$ to $r_{i+1,k}$ in $\mathcal{N}'$, as required.

Finally, as noted above, $\lambda(\mathcal{N}') \geq \lambda(\mathcal{N})$ must hold. Therefore, in view of the facts that (i) $\mathcal{N}'$ is by construction a subnetwork of $\mathcal{N}$ that does not contain any edge in $E_{i+1}$ and (ii) $\mathcal{N}'$ and $\tilde{\mathcal{N}}$ contain the same subset $H$ of edges in $E_{i+1}$, we must also have $\lambda(\tilde{\mathcal{N}}) \geq \lambda(\mathcal{N}')$. Since $\mathcal{N}$ is a minimum Manhattan network for $P_{i+1}$ and $\tilde{\Pi}$, this immediately implies that also $\mathcal{N}'$ is a minimum Manhattan network for $P_{i+1}$ and $\tilde{\Pi}$, as claimed.

We continue with a simple observation that will be used later on.

**Lemma 2.** Let $i \in \{1, \ldots, l\}$, let $\Pi' = ((A'_1, \ldots, A'_h), (B'_1, \ldots, B'_h))$ be a pair that is admissible for $P_i$ and let $\mathcal{N}$ be a minimum Manhattan network for $P_i$ and $\Pi'$. Then $\Pi'$ is made redundant by $\pi(\mathcal{N})$.

**Proof:** By definition of a minimum Manhattan network for $P_i$ and $\Pi'$, the network $\mathcal{N}$ must contain a monotone path from $v_{i,j}$ to $r_{i,k}$ for all $k \in A'_j \cup B'_j$ and all $j \in \{1, \ldots, h\}$. But this immediately implies that $A'_j \subseteq A_j(\mathcal{N})$ and $B'_j \subseteq B_j(\mathcal{N})$ holds for all $j \in \{1, \ldots, h\}$. This implies that $\pi(\mathcal{N})$ makes $\Pi'$ redundant in view of the fact that $\mathcal{N}$ is a minimum Manhattan network for $P_i$ and $\Pi'$ as well as for $P_i$ and $\pi(\mathcal{N})$.

Now we are in a position to establish the correctness of our algorithm.

**Corollary 1.** The collection $\mathcal{C}_i$ constructed by our algorithm, $i \in \{1, \ldots, l\}$, is such that

(*) $A_i := \{\pi(\mathcal{N}) : \mathcal{N} \in \mathcal{C}_i\}$ is a cover, and, for every $\Pi \in A_i$, there is a minimum Manhattan network for $P_i$ and $\Pi$ in $\mathcal{C}_i$.

**Proof:** We use induction on $i$. For $i = 1$ our algorithm considers all pairs $\Pi$ that are admissible for $P_1$ and computes a minimum Manhattan network $\mathcal{N}$ for $P_1$ and $\Pi$. This implies that (*) holds for $\mathcal{C}_1$.

So, assume that (*) has been established for some $\mathcal{C}_i$, $i \in \{1, \ldots, l-1\}$, and consider $\mathcal{C}_{i+1}$. Let $\Pi'$ be an arbitrary pair that is admissible for $P_{i+1}$. Using induction and Lemma 1, it follows that there exists a Manhattan network $\mathcal{N}$ in $\mathcal{C}_i$ and a subset $H \subseteq E_{i+1}$ such that the Manhattan network $\mathcal{N}'$ obtained by adding the edges in $H$ to $\mathcal{N}$ is a minimum Manhattan network for $P_{i+1}$ and $\Pi$. By Lemma 2, $\Pi'$ is made redundant by $\pi(\mathcal{N}')$. Since $\Pi'$ was chosen arbitrarily, this immediately implies that $A_{i+1}$ is a cover. Moreover, as the above argument includes the special case where $\Pi$ is an element of $A_{i+1}$, it also follows that, for every $\Pi \in A_{i+1}$, there is a minimum Manhattan network for $P_i$ and $\Pi$ in $\mathcal{C}_{i+1}$

**Corollary 2.** The collection $\mathcal{C}_i$ contains a minimum Manhattan network for $P_i$.

**Proof:** Let $\tilde{\mathcal{N}}$ be a minimum Manhattan network for $P_i$. Consider the canonical pair $\tilde{\Pi} := \pi(\tilde{\mathcal{N}})$ associated to $\tilde{\mathcal{N}}$. By Corollary 1 and Lemma 1, there exists a Manhattan network
network $\mathcal{N}$ in $C_{l-1}$ and a subset $H \subseteq E_l$ such that the network $\mathcal{N}'$ obtained by adding the edges in $H$ to $\mathcal{N}$ is a minimum Manhattan network for $P_l = P$ and $\Pi$. But then $\mathcal{N}'$ is also a minimum Manhattan network for $P$, as required.

In the remainder of this section we focus on bounding the run time and space for our algorithm. The key ingredient will be to describe the admissible pairs that occur in $A_i = \{\pi(\mathcal{N}) : \mathcal{N} \in C_i\}$ in such a way that a suitable upper bound on $|A_i|$ can be derived. Intuitively, for every admissible pair $\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))$ in $A_i$, we associate with each non-empty set $A_j$, $j \in \{1, \ldots, h\}$, an axis-parallel rectangle $K_j$ whose upper left corner is at $(x_1, y_h)$ and that contains precisely those among the points in $\{r_{i,k} : k \in \bigcup_{s=1}^{h} A_s\}$ for which $k \in A_j$. Similarly, we associate with each non-empty set $B_j$, $j \in \{1, \ldots, h\}$, an axis-parallel rectangle $K'_j$ with lower left corner at $(x_1, y_1)$ such that $K'_j$ contains precisely those among the points in $\{r_{i,k} : k \in \bigcup_{s=1}^{h} B_s\}$ for which $k \in B_j$.

Note that, at first glance, the smallest axis-parallel rectangle with upper left corner at $(x_1, y_h)$ that contains the point set $\{r_{i,k} : k \in A_j\}$ might appear to be a good choice for $K_j$ (cf. Lemma 3 below). Similarly, for $K'_j$ one might choose the smallest axis-parallel rectangle with lower left corner at $(x_1, y_1)$ that contains the point set $\{r_{i,k} : k \in B_j\}$. With a more careful choice of these rectangles, however, we can achieve that the $x$- and $y$-coordinates of the lower right corners of the rectangles describing the non-empty sets $A_j$, $j \in \{1, \ldots, h\}$, as well as the $x$- and $y$-coordinates of the upper right corners of the rectangles describing the non-empty sets $B_j$, $j \in \{1, \ldots, h\}$, form monotone sequences, a fact that is subsequently used to establish an upper bound on $|A_i|$. To present the construction of these sequences, next we introduce some more notation where we use some special element $\perp$ not contained in $\mathbb{R}$ to indicate that the corresponding sets $A_j$ and $B_j$, $j \in \{1, \ldots, h\}$, are empty and, therefore, have no rectangle associated with it.

So, put $X_i := \{x(r_{i,j}) : j \in R_i\}$, that is, the set of the $x$-coordinates of the rightmost point in $P_i$ on each line $L_j$. A sequence $z_1, z_2, \ldots, z_h$ of elements in $X_i \cup \{\perp\}$ is called increasing on $X_i$ if, for all $j_1, j_2 \in \{1, \ldots, h\}$ with $j_1 \leq j_2$, $z_{j_1} \neq \perp$ and $z_{j_2} \neq \perp$, we have $z_{j_1} \leq z_{j_2}$. Similarly, a sequence $z'_1, z'_2, \ldots, z'_h$ of elements in $Y \cup \{\perp\}$ is called increasing (decreasing) on $Y$ if, for all $j_1, j_2 \in \{1, \ldots, h\}$ with $j_1 \leq j_2$, $z'_{j_1} \neq \perp$ and $z'_{j_2} \neq \perp$, we have $z'_{j_1} \leq z'_{j_2}$ ($z'_{j_1} \geq z'_{j_2}$). Consider, for example, the point set $P_5$ depicted in Figure 4(a). We have $X_5 = \{x_2, x_3, x_4, x_5\}$ and $\perp, x_4, x_5, \perp$ is an increasing sequence on it.

In addition, we define a 6-tuple $\Phi = (S_1, S'_1, S_2, S'_2, A, B)$ on $X_i$ and $Y$, consisting of two increasing sequences $S_1 = a_1, \ldots, a_h$ and $S_2 = b_1, \ldots, b_h$ on $X_i$, an increasing sequence $S'_1 = a'_1, \ldots, a'_h$ on $Y$, a decreasing sequence $S'_2 = b'_1, \ldots, b'_h$ on $Y$ and two subsets $A$ and $B$ of $R_i$, to be compatible if $a_j = \perp \iff a'_j = \perp$ and $b_j = \perp \iff b'_j = \perp$ hold for all $j \in \{1, \ldots, h\}$. Note that the purpose of the sequences $S_1$ and $S'_1$ is to list the $x$- and $y$-coordinates of the lower right corner of the rectangle $K_j$ associated with any non-empty set $A_j$, $j \in \{1, \ldots, h\}$, in a pair $\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h))$ that is admissible for $P_i$. Similarly, $S_2$ and $S'_2$ list the $x$- and $y$-coordinates of the upper right corner of the rectangle $K'_j$ associated with any non-empty set $B_j$, $j \in \{1, \ldots, h\}$, in $\Pi$. Moreover, $A$ and $B$ represent the sets $\bigcup_{k=1}^{h} A_k$ and $\bigcup_{k=1}^{h} B_k$, respectively. Referring again to Figure 4(a), putting $S_1 = \perp, x_4, x_5, \perp$, $S'_1 = \perp, y_2, y_3, \perp$, $S_2 = x_2, x_4, x_5, \perp$, $S'_2 = y_4, y_3, y_1, \perp$, $A = \{1, 2, 3\}$ and $B = \{2, 4\}$, we obtain an example of a compatible 6-tuple $\Phi$ on $\{x_2, x_3, x_4, x_5\}$ and $\{y_1, y_2, y_3, y_4\}$.
and otherwise. It follows immediately from the definition that \( \pi \) captures the information in a given admissible pair, it might be helpful to first describe how, conversely, compatible 6-tuples on \( X \) and \( Y \) give rise to pairs that are admissible for \( P_1 \). More specifically, for any compatible 6-tuple \( \Phi \) on \( X \) and \( Y \), define the pair \( \pi(\Phi) = ((A_1, \ldots, A_h), (B_1, \ldots, B_h)) \) by putting, for all \( j \in \{1, \ldots, h\} \), \( A_j := \emptyset \) in case \( a_j = a_j' = \perp \) and \( A_j := \{ k \in A : x(r_{i,k}) \leq a_j, \; y(r_{i,k}) \geq a_j' \}, \; k \geq j \} \) otherwise and, similarly, \( B_{h-j+1} := \emptyset \) in case \( b_j = b_j' = \perp \) and \( B_{h-j+1} := \{ k \in B : x(r_{i,k}) \leq b_j, \; y(r_{i,k}) \leq b_j' \}, \; k < h - j + 1 \} \) otherwise. It follows immediately from the definition that \( \pi(\Phi) \) is admissible for \( P_1 \). To illustrate the definition of \( \pi(\Phi) \), consider the compatible 6-tuple \( \Phi \) on \( \{x_2, x_3, x_4, x_5\} \) and \( \{y_1, y_2, y_3, y_4\} \) constructed at the end of the previous paragraph. For this \( \Phi \) we obtain \( \pi(\Phi) = ((\emptyset, \{2\}, \emptyset), (\emptyset, \emptyset, \{2\}, \emptyset)) \). As illustrated in Figure 4(b) and (c), the sets \( A_j \) and \( B_j \), \( j \in \{1, \ldots, 4\} \), are indeed represented by axis-parallel rectangles. Consider, for example, the set \( B_4 \). The first elements of \( S_2 \) and \( S'_2 \) determine the \( x \)- and \( y \)-coordinate, respectively, of the upper right corner of the corresponding rectangle \( K'_4 \). As mentioned above, the lower left corner of \( K'_4 \) is at \((x_1, y_1)\). Then (cf. Figure 4(b)), among the points \( r_{5,k}, k \in R_5 \), only \( r_{5,1} \) lies in \( K'_4 \) and we even have \( 1 < 4 \). But we have \( 1 \notin B = \{2, 4\} \). Therefore, we obtain \( B_4 = \emptyset \). Similarly, we obtain \( B_3 = \{2\} \) (cf. Figure 4(c)).

In the next lemma, using the construction alluded to above, we establish that, for any Manhattan network \( \mathcal{N} \) for \( P_i \), \( i \in \{1, \ldots, l\} \), and any non-empty set \( A_j \), \( j \in \{1, \ldots, h\} \), in the canonical admissible pair \( ((A_1, \ldots, A_h), (B_1, \ldots, B_h)) = \pi(\mathcal{N}) \), the elements in \( A_j \) can be described by some rectangle. The construction of some rectangle describing the elements in any non-empty set \( B_j \), \( j \in \{1, \ldots, h\} \), in \( \pi(\mathcal{N}) \) is completely analogous.

**Lemma 3.** Let \( \mathcal{N} \) be a Manhattan network for \( P_i \) and \( ((A_1, \ldots, A_h), (B_1, \ldots, B_h)) = \pi(\mathcal{N}) \) be the canonical pair that is admissible for \( P_i \). Put \( A := \bigcup_{k=1}^h A_k \) and let \( j \in \{1, \ldots, h\} \) be such that \( A_j \neq \emptyset \). In addition, let \( k_1 \in A_j \) be such that \( x(r_{i,k_1}) = \max\{x(r_{i,k}) : k \in A_j\} \) and put \( a_j := x(r_{i,k_1}) \). Similarly, let \( k_2 \in A_j \) be such that \( y(r_{i,k_2}) := \min\{y(r_{i,k}) : k \in A_j\} \) and put \( a'_j := y(r_{i,k_2}) \). Then, for every \( k \in A \) with \( x(r_{i,k}) \leq a_j \) and \( y(r_{i,k}) \geq a'_j \), a monotone path in \( \mathcal{N} \) from \( v_{i,j} \) to \( r_{i,k} \) exists, that is, \( k \in A_j \).

**Proof:** To illustrate the construction described in the lemma, consider, for example, the
Figure 5: (a) The rectangle $K_1$ describing the set $A_1$ constructed in the proof of Lemma 3 for the Manhattan network in Figure 1(c). The points $r_{6,k}$ with $k \in A_1$ are marked as large black dots. (b) A possible rectangle to describe the set $A_3$, but it does not yield an increasing sequence $S_1$. The points $r_{6,k}$ with $k \in A_3$ are again marked as large black dots. (c) Stretching the rectangle in (b) to the right, we obtain the rectangle $K_3$ which also describes the set $A_3$ and, in addition, yields an increasing sequence $S_1$.

Manhattan network for $P_6$ depicted in Figure 1(c). We have $A = \{1, 2, 3, 4\}$ and, for $j = 1$, we obtain $k_1 = 1$ and $k_2 = 1$. This yields $(x_6, y_1)$ as the lower right corner of the rectangle $K_1$ in this case, as depicted in Figure 5(a).

Now, to prove the lemma, assume for a contradiction that there exists some $k \in A$ with $x(r_{i,k}) \leq a_j$ and $y(r_{i,k}) \geq a'_j$ but $k \notin A_j$. We cannot have $x(r_{i,k}) \leq x(r_{i,k_2})$ in view of the fact that there must exist a monotone path from $r_{i,k_2}$ to $r_{i,k}$ in $\mathcal{N}$, since $\mathcal{N}$ is a Manhattan network for $P_i$, and the concatenation of a monotone path from $v_{i,j}$ to $r_{i,k_2}$ and a monotone path from $r_{i,k_2}$ to $r_{i,k}$ yields a monotone path from $v_{i,j}$ to $r_{i,k}$ in $\mathcal{N}$. Similarly, we obtain a monotone path from $v_{i,j}$ to $r_{i,k}$ in case $y(r_{i,k}) \geq y(r_{i,k_1})$. So it remains to consider the case that $r_{i,k}$ is contained in the axis-parallel rectangle with lower left corner at $r_{i,k_2}$ and upper right corner at $r_{i,k_1}$. Then, since $k \in A \setminus A_j$, there must exist some $j^* \in \{1, \ldots, h\}$ such that $k \in A_{j^*}$. Hence, $\mathcal{N}$ must contain a monotone path $p$ from $v_{i,j^*}$ to $r_{i,k}$. We distinguish two cases:

**Case 1:** $j^* < j$. The situation is depicted in Figure 6(a). The path $p$ has at least one vertex in common with any monotone path from $v_{i,j}$ to $r_{i,k_2}$. Therefore, $\mathcal{N}$ must also contain a monotone path from $v_{i,j}$ to $r_{i,k}$, a contradiction.

**Case 2:** $j^* > j$. The situation is depicted in Figure 6(b). Similarly to the previous case, the path $p$ has at least one vertex in common with any monotone path from $v_{i,j}$ to $r_{i,k_1}$. Therefore, also in this case $\mathcal{N}$ must contain a monotone path from $v_{i,j}$ to $r_{i,k}$, a contradiction.

The following lemma establishes that the number of compatible 6-tuples on $X_i$ and $Y$ is an upper bound on $|C_i|$.

**Lemma 4.** There is an injective map $\varphi$ from the set $A_i$ into the set of compatible 6-tuples on $X_i$ and $Y$ such that $\pi(\varphi(\Pi)) = \Pi$ holds for all $\Pi \in A_i$. 
sequences $S$ is always possible to find some suitable rectangle where $\varphi(\Pi) = (S_1, S_1', S_2, S_2', A, B)$. To construct a suitable axis-parallel rectangle $K_j$ with upper left corner at $(x_1, y_h)$ that contains precisely those points in $\{r_{i,k} : k \in A\}$ for which $k \in A_{j_1}$, we use the construction described in Lemma 3. This will place the lower right corner of $K_j$ at a suitable point. Recall that, to construct the coordinates of this point, we let $k_1 \in A_{j_1}$ be such that $x(r_{i,k_1}) = \max\{x(r_{i,k}) : k \in A_{j_1}\}$ and put $a_{j_1} := x(r_{i,k_1})$. Similarly, we let $k_2 \in A_{j_1}$ be such that $y(r_{i,k_2}) := \min\{y(r_{i,k}) : k \in A_{j_1}\}$ and put $a'_{j_1} := y(r_{i,k_2})$.

The basic idea for continuing our construction of $S_1$ and $S_1'$ is to next consider the smallest index $j_2 \in \{j_1 + 1, \ldots, h\}$ with $A_{j_2} \neq \emptyset$, if such an index exists. Ideally, we would like to use again Lemma 3 to construct a suitable rectangle $K_{j_2}$ describing the set $A_{j_2}$. This is, however, not always possible as illustrated in Figure 5(b). More specifically, the $x$- and $y$-coordinate of the lower right corner of the so constructed rectangle need not yield sequences $S_1$ and $S_1'$ that are increasing. In the remainder of the proof we will show that it is always possible to find some suitable rectangle $K_{j_2}$ such that the resulting sequences $S_1$ and $S_1'$ are guaranteed to be increasing, as indicated in Figure 5(c).

To describe the construction in general, assume that we have already found values $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$, $j_1 < j_2 < \cdots < j_s$, up to some index $j_s \in \{1, \ldots, h\}$, where $\{j_1, j_2, \ldots, j_s\}$ consists of those indices $j \in \{1, 2, \ldots, j_s\}$ with $A_{j} \neq \emptyset$, that satisfy the following properties:

(a) $a_{j_1} \leq a_{j_2} \leq \cdots \leq a_{j_s}$ and $a'_{j_1} \leq a'_{j_2} \leq \cdots \leq a'_{j_s}$, that is, we have increasing prefixes of $S_1$ and $S_1'$.

(b) The set $A_j$ equals, for every $j \in \{j_1, j_2, \ldots, j_s\}$, the set of those indices $k \in A$ with $x(r_{i,k}) \leq a_j$ and $y(r_{i,k}) \geq a'_j$, that is, we have found a suitable rectangle $K_j$ with upper left corner at $(x_1, y_h)$ and lower right corner at $(a_j, a'_j)$ that describes $A_j$. 

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Figure 6: This illustrates the two cases considered in the proof of Lemma 3. In both cases, the two monotone paths depicted induce a monotone path from $v_{i,j}$ to $r_{i,k}$. (a) Case 1 (b) Case 2.

Proof: To describe the map $\varphi$, we will only show how, for an arbitrary pair $\Pi \in A_i$, a suitable increasing sequence $S_1 = a_1, \ldots, a_h$ on $X_i$, an increasing sequence $S_1' = a'_1, \ldots, a'_h$ on $Y$ and a set $A \subseteq R_i$ can be constructed. Suitable sequences $S_2$ and $S_2'$ as well as a set $B$ can then be constructed in a completely analogous way, yielding together the compatible 6-tuple $\Phi = (S_1, S_1', S_2, S_2', A, B) = \varphi(\Pi)$.

Let $N$ denote the Manhattan network in $C_i$ with $\Pi = ((A_1, \ldots, A_h), (B_1, \ldots, B_h)) := \pi(N)$. We put, for every $j \in \{1, \ldots, h\}$ with $A_j = \emptyset$, $a_j = a'_j := \bot$ and put $A := \bigcup_{j=1}^h A_j$. If $A = \emptyset$, this finishes our construction. Otherwise, consider the smallest index $j_1 \in \{1, \ldots, h\}$ with $A_{j_1} \neq \emptyset$. To construct a suitable axis-parallel rectangle $K_{j_1}$ with upper left corner at $(x_1, y_h)$ that contains precisely those points in $\{r_{i,k} : k \in A\}$ for which $k \in A_{j_1}$, holds, we use the construction described in Lemma 3. This will place the lower right corner of $K_{j_1}$ at a suitable point. Recall that, to construct the coordinates of this point, we let $k_1 \in A_{j_1}$ be such that $x(r_{i,k_1}) = \max\{x(r_{i,k}) : k \in A_{j_1}\}$ and put $a_{j_1} := x(r_{i,k_1})$. Similarly, we let $k_2 \in A_{j_1}$ be such that $y(r_{i,k_2}) := \min\{y(r_{i,k}) : k \in A_{j_1}\}$ and put $a'_{j_1} := y(r_{i,k_2})$.

The basic idea for continuing our construction of $S_1$ and $S_1'$ is to next consider the smallest index $j_2 \in \{j_1 + 1, \ldots, h\}$ with $A_{j_2} \neq \emptyset$, if such an index exists. Ideally, we would like to use again Lemma 3 to construct a suitable rectangle $K_{j_2}$ describing the set $A_{j_2}$. This is, however, not always possible as illustrated in Figure 5(b). More specifically, the $x$- and $y$-coordinate of the lower right corner of the so constructed rectangle need not yield sequences $S_1$ and $S_1'$ that are increasing. In the remainder of the proof we will show that it is always possible to find some suitable rectangle $K_{j_2}$ such that the resulting sequences $S_1$ and $S_1'$ are guaranteed to be increasing, as indicated in Figure 5(c).

To describe the construction in general, assume that we have already found values $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$, $j_1 < j_2 < \cdots < j_s$, up to some index $j_s \in \{1, \ldots, h\}$, where $\{j_1, j_2, \ldots, j_s\}$ consists of those indices $j \in \{1, 2, \ldots, j_s\}$ with $A_{j} \neq \emptyset$, that satisfy the following properties: 

(a) $a_{j_1} \leq a_{j_2} \leq \cdots \leq a_{j_s}$ and $a'_{j_1} \leq a'_{j_2} \leq \cdots \leq a'_{j_s}$, that is, we have increasing prefixes of $S_1$ and $S_1'$.

(b) The set $A_j$ equals, for every $j \in \{j_1, j_2, \ldots, j_s\}$, the set of those indices $k \in A$ with $x(r_{i,k}) \leq a_j$ and $y(r_{i,k}) \geq a'_j$, that is, we have found a suitable rectangle $K_j$ with upper left corner at $(x_1, y_h)$ and lower right corner at $(a_j, a'_j)$ that describes $A_j$. 

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implies that Properties (a)-(c) hold. It is also not hard to check that Properties (a)-(c) hold. So assume that there exists some $j \in \{j_0+1, j_0+2, \ldots, h\}$ with $A_j \neq \emptyset$ and let $j_{s+1} \in \{j_0+1, j_0+2, \ldots, h\}$ be such that, after adjusting some of the values $a_{j_0}, a_{j_0+1}, \ldots, a_{j_{s+1}}$ if necessary, the resulting sequences $a_{j_0}, a_{j_0+1}, \ldots, a_{j_{s+1}}$ again satisfy Properties (a)-(c) (with $s$ replaced by $s+1$). To this end, similarly as for $j_1$, let $k_1 \in A_{j_{s+1}}$ be such that $x(r_{i,k_1}) = \max \{x(r_{i,k}) : k \in A_{j_{s+1}} \}$ and put $q_1 := r_{i,k_1}$. Similarly, let $k_2 \in A_{j_{s+1}}$ be such that $y(r_{i,k_2}) = \min \{y(r_{i,k}) : k \in A_{j_{s+1}} \}$ and put $q_2 := r_{i,k_2}$. By Lemma 3 the rectangle $K$ with upper left corner at $(x_1, y_1)$ and lower right corner at $(x(q_1), y(q_2))$ describes the set $A_{j_{s+1}}$. We divide our argument into four cases.

Case 1: $a_{j_s} \leq x(q_1)$ and $a'_{j_s} \leq y(q_2)$. Then, putting $a_{j_{s+1}} := x(q_1)$ and $a'_{j_{s+1}} := y(q_2)$, it is not hard to check that Properties (a)-(c) hold.

Case 2: $a_{j_s} > x(q_1)$ and $a'_{j_s} \leq y(q_2)$. The situation is depicted in Figure 7(a). We claim that there is no $k \in A$ with $x(q_1) < x(r_{i,k}) \leq a_{j_s}$ and $y(q_2) \leq y(r_{i,k})$, that is, no $r_{i,k}$ with $k \in A$ lies in the region indicated by shading in Figure 7(a). To show this, assume for a contradiction that there is such a $k \in A$. Note that, since Property (b) holds for $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$, there exists a monotone path from $v_{i,j_s}$ to $r_{i,k}$ in $N$. This path has at least one vertex in common with any monotone path in $N$ from $v_{i,j_{s+1}}$ to $q_2$, implying that there also exists a monotone path from $v_{i,j_{s+1}}$ to $r_{i,k}$ (cf. Figure 7(b)). But this contradicts the choice of $k_1$ above. Hence, putting $a_{j_{s+1}} := a_{j_s}$ and $a'_{j_{s+1}} := y(q_2)$, that is, stretching the rectangle $K$ to the right, it is not hard to check that Properties (a)-(c) hold.

Case 3: $a_{j_s} \leq x(q_1)$ and $a'_{j_s} > y(q_2)$. The situation is depicted in Figure 8(a). Let $j^* \in \{j_1, j_2, \ldots, j_s\}$ be such that $a_{j^*} = \max \{x(r_{i,k}) : k \in A_{j^*} \}$. Such an index $j^*$ must exist, since Property (c) holds for $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$. First note that there is no $k \in A$ with $x(r_{i,k}) \leq a_{j^*}$ and $y(q_2) < y(r_{i,k}) < a'_{j^*}$. Assume for a contradiction that such a $k$ exists. Then, in view of the fact that $N$ must contain a monotone path from $v_{i,j^*}$ to a point $r_{i,t}$, $t \in A_{j^*}$, with maximum $x$-coordinate as well as a monotone path from $v_{i,j_{s+1}}$.
to $r_{i,k}$, it follows that $\mathcal{N}$ also contains a monotone path from $v_{i,j^*}$ to $r_{i,k}$ (cf. Figure 8(b)), contradicting that Property (b) holds for $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$.

Similarly, there is no $k \in A$ with $x(r_{i,k}) \leq a_{j_s}$ and $\max\{y(q_2), a'_{j_s}\} \leq y(r_{i,k}) < a'_{j_s}$. Assume again for a contradiction that such a $k$ exists. Then, in view of the fact that $\mathcal{N}$ must contain monotone paths from $v_{i,j^*}$ and $v_{i,j_s+1}$ to $r_{i,k}$ as well as a monotone path from $v_{i,j_s}$ to some point $r_{i,t}$ with $t \in A_{j_s}$, it follows that $\mathcal{N}$ also contains a monotone path from $v_{i,j_s}$ to $r_{i,k}$ (cf. Figure 8(c)), again contradicting that Property (b) holds for $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$. As a consequence, putting $a_{j_s+1} := x(q_1)$, $a'_{j_s+1} := y(q_2)$ and, for every $j \in \{j_1, j_2, \ldots, j_s\}$ with $y(q_2) < a'_j$, $a'_j := y(q_2)$, that is, stretching the rectangle $K_j$ downwards, it is again not hard to check that Properties (a)-(c) hold.

**Case 4:** $a_{j_s} > x(q_1)$ and $a'_{j_s} > y(q_2)$. The situation is depicted in Figure 9(a). As in Case 3, consider an index $j^* \in \{j_1, j_2, \ldots, j_s\}$ such that $a_{j_s} = \max\{x(r_{i,k}) : k \in A_{j^*}\}$. Then, in case $a'_{j^*} > y(q_2)$, there would be a monotone path from $v_{i,j^*}$ to $q_2$ in $\mathcal{N}$, in view of the fact that $\mathcal{N}$ must contain a monotone path from $v_{i,j^*}$ to a point $r_{i,t}$, $t \in A_{j^*}$, with maximum $x$-coordinate and also a monotone path from $v_{i,j_s+1}$ to $q_2$ (cf. Figure 9(b)). But this contradicts the fact that Property (b) holds for $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$.

And in case $a'_{j^*} \leq y(q_2)$, there would be a monotone path from $v_{i,j_s}$ to $q_2$ in $\mathcal{N}$ in view of the fact that $\mathcal{N}$ must contain monotone paths from $v_{i,j^*}$ and $v_{i,j_s+1}$ to $q_2$ and also a monotone path from $v_{i,j_s}$ to some point $r_{i,t}$ with $t \in A_{j_s}$ (cf. Figure 9(c)). This also contradicts the fact that Property (b) holds for $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$ and $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_s}$. Hence Case 4 never applies. This finishes the case analysis.

To conclude the proof of the lemma, note that by construction we have indeed $\pi(\varphi(\Pi)) = \Pi$, implying that $\varphi$ is injective.

We conclude this section summarizing the main result of this paper.

**Theorem 1.** A minimum Manhattan network for $P$ can be computed in $O^*(2^{14h})$ time and $O^*(2^{12h})$ space.

**Proof:** The correctness of our algorithm was established in Corollary 2. To bound the space used by our algorithm, it suffices to give an upper bound on $|C_i| = |A_i|$, $i \in \{1, \ldots, l\}$. For $i = 1$, while it is convenient in the pseudocode of the algorithm to write that we loop through all pairs $\Pi$ that are admissible for $P_1$, we have already observed in the description
of the algorithm in the text that a minimum Manhattan network for $P_1$ and $\Pi$ is uniquely determined and, therefore, the number of Manhattan networks that it suffices to consider for $i = 1$ can be bounded by $O(h^2)$: we simply choose the vertex with minimum and maximum $y$-coordinate in the network.

For $i > 1$, we rely on Lemma 4 and give an upper bound on the number of compatible 6-tuples on $X_i$ and $Y$. We claim that the number of these 6-tuples is in $O^*(2^{12h})$. To show the claim, we first bound the number of pairs of sequences $S_1$ and $S'_1$ that can occur in a compatible 6-tuple $T = (S_1, S'_1, S_2, S'_2, A, B)$ on $X_i$ and $Y$. Suppose there are precisely $r$ entries in $S_1$ and $S'_1$ which equal $\bot$. The remaining entries of $S_1$ and $S'_1$, since they are sorted, correspond to a submultiset of $X_i$ and $Y$, respectively, of size $h - r$. In view of $|X_i| \leq |Y| = h$, this implies that there are no more than $h \cdot 2^h \cdot 2^{2h} = h \cdot 2^{5h}$ pairs $S_1$ and $S'_1$. An analogous argument yields the same bound on the number of pairs $S_2$ and $S'_2$. Finally, the number of subsets $A$ and $B$ is clearly bounded by $2^h$ each, implying that there are $O^*(2^{12h})$ compatible 6-tuples on $X_i$ and $Y$, as claimed.

Finally, to bound the run time of our algorithm, note that there are at most $2^{2h}$ subsets $H$ of $E_{i+1}$ considered by our algorithm. Hence, using the fact that there are $O^*(2^{12h})$ Manhattan networks in $C_i$ established above, it follows that the run time is in $O^*(2^{14h})$.

4 Concluding remarks

While here we were mainly concerned with establishing the existence of a fixed-parameter algorithm for MMN with a run time in $O^*(c^h)$ for some constant $c$, it could be interesting to try and refine the approach in order to design a fixed-parameter algorithm for MMN that is competitive to the existing exact algorithm for MMN based on integer linear programming. As for many other natural parameterizations, however, a subexponential run time in terms of $h$ is ruled out under the exponential time hypothesis. This follows from the fact that in the reduction from 3-SAT to MMN presented in [8] a Boolean formula $F$ with $m$ literals is transformed into an input point set $P(F)$ for MMN that is contained in $O(m)$ horizontal straight lines. Hence, any algorithm for MMN running in subexponential time with respect to the parameter $h$ would yield a subexponential time algorithm for 3SAT with respect to the parameter $m.$
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References


