POINTS WITH LARGE QUADRANT DEPTH

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Abstract.

Given a set P of points in the plane we are interested in points that are 'deep' in the set in the sense that they have two opposite quadrants both containing many points of P. We deal with an extremal version of this problem. A pair (a, b) of numbers is *admissible* if every point set P contains a point $p \in P$ that determines a pair (Q, Q^{op}) of opposite quadrants, such that Q contains at least an a-fraction and Q^{op} contains at least a b-fraction of the points of P. We provide a complete description of the set $\mathcal{F}$ of all admissible pairs (a, b). This amounts to identifying three line segments and a point on the boundary of $\mathcal{F}$.

In higher dimensions we study the maximum a, such that (a, a) is opposite-orthant admissible. In dimension d we show that $1/(2\gamma) \leq a \leq 1/\gamma$ for $\gamma = 2^{2^{d-1}} 2^{d-1}$.

Finally we deal with a variant of the problem where the opposite pairs of orthants need not be determined by a point in P. Again we are interested in values a, such that all subsets P in $\mathbb{R}^d$ admit a pair (O, O^{op}) of opposite orthants both containing at least an a-fraction of the points. The maximum such value is $a = 1/2^d$. Generalizations of the problem are also discussed.

1 Introduction

A point $p = (p_1, p_2)$ in the plane defines four quadrants $Q_1(p), Q_2(p), Q_3(p)$, and $Q_4(p)$ centered at p, each being the intersection of two halfspaces defined by one horizontal and one vertical line through p. As usual the quadrants are numbered in counterclockwise order starting from $Q_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq p_1 \text{ and } x_2 \geq p_2\}$. There are two pairs of opposite quadrants (Q_1, Q_3) and (Q_2, Q_4). We write Q^{op} to denote the quadrant opposite to Q, e.g., $Q_3^{\mathsf{op}} = Q_1$. In the extended abstract of this article, which appeared in the 2010 SOCG Conference Proceedings [ABF+], we have considered the quadrants to be open. However, our results hold for closed quadrants as well. The closed quadrants have the advantage that we need no assumption about general position.

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For a given set P of points we ask for points $p \in P$ that are 'deep' in the set in the sense that p has two opposite quadrants both containing many points of P. A pair (Q, Q^{op}) of opposite quadrants centered at a point $p \in P$ is called (a, b)-*admissible* if $|Q \cap P| \geq a(n-1)$ and $|Q^{\mathsf{op}} \cap P| \geq b(n-1)$. We call a pair (a, b) *admissible* if *every* finite point set P contains a point with two opposite quadrants that are (a, b)-admissible. Thus, (a, b) is *not* admissible if there exists a point set P in which no point has two opposite quadrants that are (a, b)-admissible. Indeed it follows from our proofs that if (a, b) is not admissible, then there exists a number N, such that for all $n \geq N$ there is an n-element set P in which no point has two opposite quadrants that are (a, b)-admissible.

Brönnimann, Lenchner, and Pach [BLP] define the notion of *opposite-quadrant depth* for point sets in the plane as the maximum a, such that (a, a) is admissible. They prove that every set of points in the plane has opposite-quadrant depth at least $\frac{1}{8}$. We give a new and simpler proof of this result below, Theorem 1.1.

In Section 2 we provide a complete description of the set $\mathcal{F}$ of all admissible pairs. The shape of $\mathcal{F}$ turns out to be surprisingly complicated (see Figure 3).

In Section 3 we ask for the maximum a, such that (a, a) is admissible in higher dimensions. In dimension d we obtain upper and lower bounds that differ by a factor of 2. In Section 4 we discuss further generalizations.

The notion of opposite-quadrant depth, resp. opposite-orthant depth, is related to *centerpoints* and some measures of statistical depth, such as *hyperplane depth*. We refer to [Ede] for information on centerpoints and to [LPS] for statistical depth. Brönnimann et al. [BLP] also mention a connection with conflict-free colorings. Related notions of *depth* have been studied e.g. in [BPZ1] and [BPZ2].

As a warm up and for the purpose of introducing some convenient notation we now reprove the main result from [BLP].

Theorem 1.1 ([BLP])**.**

(1) *Any set P of n points in the plane has opposite-quadrant depth at least $\frac{1}{8}$.*

(2) *If P is in convex position, then it has opposite-quadrant depth at least $\frac{1}{4}$.*

Before starting with the proof let us introduce the following convenient notation. Given a set P of n points, the *weight* of a subset A of the plane is $\omega(A) = \frac{|A \cap P|}{n-1}$. In terms of weights, a pair (Q, Q^{op}) of opposite quadrants at a point $p \in P$ is (a, b)-admissible if and only if $\omega(Q) \geq a$ and $\omega(Q^{\mathsf{op}}) \geq b$.

In many cases we will choose a subset P' of P of some specified weight. When we choose a set $P' \subseteq P$ of weight a we mean that P' contains exactly $\lfloor a(n-1) \rfloor$ points from P. This way, the weight of P' may be less than a, but the addition of any point would result in a weight which is at least a. The additional point will correspond to that point in P that determines an admissible pair of quadrants.

Theorem 1.1 is an easy consequence of the proof of the following lemma.

Lemma 1.2. *Every set P in the plane contains a point p, such that*

$$\min\big(\omega(Q_1(p)), \omega(Q_3(p))\big) + \min\big(\omega(Q_2(p)), \omega(Q_4(p))\big) \geq \frac{1}{4}.$$

Proof. Given the point set P, choose the sets P_L, P_R, P_B, and P_T of weight $\frac{1}{4}$ each consisting of the first points in P from the left, right, bottom, and top, respectively (see Figure 1). That is, we choose the $\lfloor \frac{1}{4}(n-1) \rfloor$ points from P with the smallest x_1-values, the largest x_1-values, the smallest x_2-values, and the largest x_2-values, respectively.

It follows that $P' = P \setminus (P_L \cup P_R \cup P_B \cup P_T) \neq \emptyset$ and we claim that every point in P' has the desired property. Let p be such a point and assume that $\min\big(\omega(Q_1(p)), \omega(Q_3(p))\big) = s = \omega(Q_1(p))$. Since P_T is contained in $Q_1(p) \cup Q_2(p)$ it follows that $\omega(Q_2(p)) \geq \frac{1}{4} - s$. Considering P_R we obtain $\omega(Q_4(p)) \geq \frac{1}{4} - s$. Consequently $\min\big(\omega(Q_1(p)), \omega(Q_3(p))\big) = s$ and $\min\big(\omega(Q_2(p)), \omega(Q_4(p))\big) \geq \frac{1}{4} - s$, which proves the lemma. □

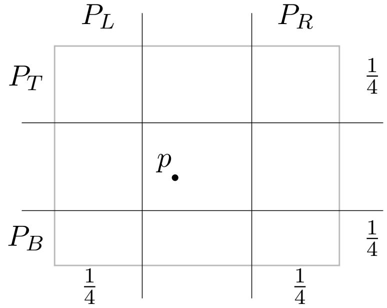

Figure 1: An illustration of the proof of Lemma 1.2.

Proof of Theorem 1.1. For part one of the theorem it is enough to observe that either s or $\frac{1}{4} - s$ is at least $\frac{1}{8}$. For the second part note that if P is in convex position one of the four quadrants of p is empty. Therefore, one of the two minima in the lemma is zero and the other minimum is at least $\frac{1}{4}$. □

It is easy to see that the second part of Theorem 1.1 is best possible by taking P to be the set of vertices of a regular n-gon. In [BLP] it is shown that the first part of Theorem 1.1 is best possible for arbitrarily large values of n of the form $n = 4 \cdot 3^k$. The example in Figure 2 shows a simple construction that proves that the first part of Theorem 1.1 cannot be improved, i.e., we show that for every $\varepsilon > 0$ there exists $N(\varepsilon)$, such that for all $n \geq N(\varepsilon)$ there is an n-element set P with opposite-quadrant depth less than $\frac{1}{8} + \varepsilon$.

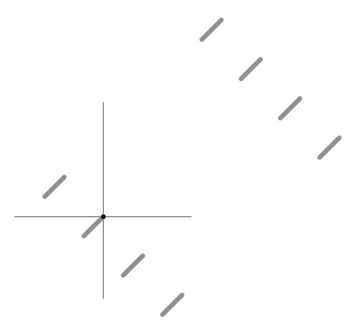

Figure 2: Opposite-quadrant depth at most $\frac{1}{8}$.

Fix $\varepsilon > 0$ and $n \geq \frac{17}{8\varepsilon} + 1$. To describe the example we identify a point set in the plane with the induced dominance order, that is, we say $(x_1, x_2) \preceq (y_1, y_2)$ if $x_1 \leq y_1$ and $x_2 \leq y_2$. Based on this order we can talk about chains and antichains of a point set. The example of Figure 2 consists of eight chains each of either $\lfloor \frac{n}{8} \rfloor$ or $\lceil \frac{n}{8} \rceil$ points. In the figure the chains are represented by gray segments.

Now $n \geq \frac{17}{8\varepsilon} + 1$ is equivalent to $(\frac{1}{8} + \varepsilon)(n-1) \geq \frac{n}{8} + 2$, which is strictly more than one point together with the points in one chain. Therefore, if a quadrant at p contains, besides p, no more than one chain, then its weight is less than $\frac{1}{8} + \varepsilon$.

The eight chains come in two groups of four chains each. One group is the set of weight $\frac{1}{2}$ consisting of the first points in P from the left (and the bottom) and the other group of weight $\frac{1}{2}$ consists of the first points in P from the right (and the top).

We consider two cases: First for $p \in P$ look at the pair $(Q_2(p), Q_4(p))$. The chains in each group are arranged in such a way that $Q_2(p)$ and $Q_4(p)$ both contain p and an integral number of chains. But there is no point p with $Q_2(p)$ *and* $Q_4(p)$ each containing two chains.

Finally, for every $p \in P$ either $Q_1(p)$ or $Q_3(p)$ contains no more than one chain. Hence every point has two adjacent quadrants with weight less than $\frac{1}{8} + \varepsilon$.

2 The Set of Admissible Pairs

Recall that a pair $(a, b) \in [0, 1]^2$ is called *admissible* if every finite point set P in the plane contains a point $p \in P$, such that there is a quadrant Q centered at p with $\omega(Q) \geq a$ and $\omega(Q^{\mathsf{op}}) \geq b$. For pairs $(a, b), (a', b')$ in $[0, 1]^2$ we write $(a, b) \preceq (a', b')$ if $a \leq a'$ and $b \leq b'$.

In this section we provide a full characterization of the set $\mathcal{F}$ of all admissible pairs. We begin with some easy observations.

- $\mathcal{F}$ is symmetric in the sense that if $(a, b) \in \mathcal{F}$, then also $(b, a) \in \mathcal{F}$.
- $\mathcal{F}$ is also monotone decreasing, that is, if $(a, b) \in \mathcal{F}$ and $(0, 0) \preceq (a', b') \preceq (a, b)$ then $(a', b') \in \mathcal{F}$.

Figure 3 depicts the half of the set $\mathcal{F}$ where $a \geq b$, the other half is obtained by reflection about the diagonal line $a = b$. In our analysis, we will determine the part of the boundary of $\mathcal{F}$ shown in the figure. That is, we will always assume that $a \geq b$.

Theorem 1.1 shows that $(\frac{1}{8}, \frac{1}{8}) \in \mathcal{F}$ and the example from Figure 2 implies that if both a and b are greater than $\frac{1}{8}$, then $(a, b) \notin \mathcal{F}$.

Though the next proposition does not really contribute to the boundary of $\mathcal{F}$ it provides a good first approximation to the set.

Proposition 2.1. *Every pair (a, b) with $3a + 5b \leq 1$ is in $\mathcal{F}$.*

Proof. First note that we only have to prove that every pair (a, b) with $3a + 5b = 1$ is in $\mathcal{F}$. The proposition then follows for all pairs $(0, 0) \preceq (a', b') \preceq (a, b)$.

So given a set P of points and a pair (a, b) with $3a + 5b = 1$, choose the set A of weight $2a$ consisting of the first points in P from the left and the set Z of weight $a + b$ consisting of the first points in P from the right. Consider the horizontal median point p in the strip S between the sets A and Z. From the assumption it follows that $\omega(S) \geq 4b$ (see Figure 4).

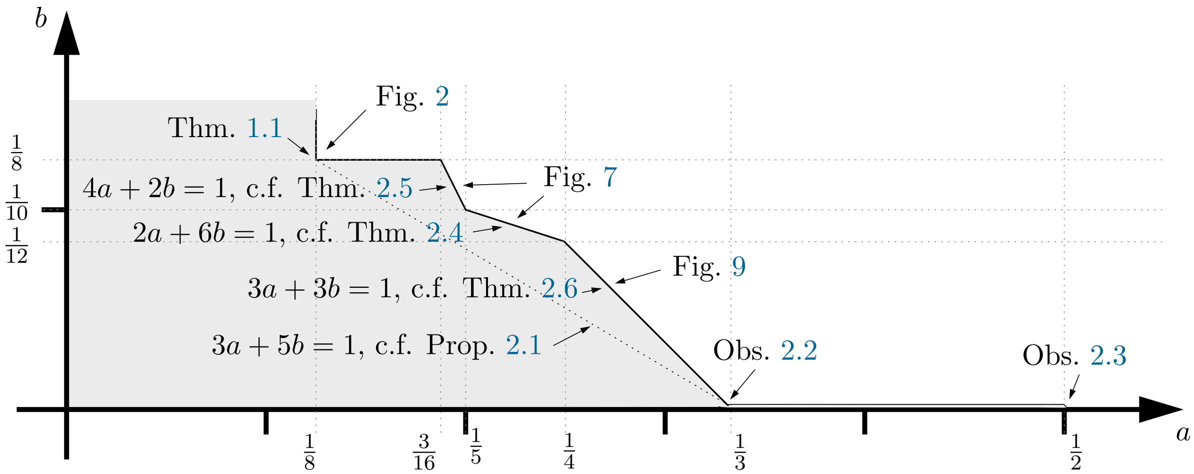

Figure 3: The shape of the set $\mathcal{F}$ of admissible pairs. References show where to find the proofs and constructions.

From $\omega(A) \geq 2a$ it follows that one of the left quadrants of p has weight at least a (w.l.o.g. the upper one $Q_2(p)$). If $\omega(Q_4(p)) \geq b$ we are done. Hence, we may assume that $\omega(Q_4(p)) < b$. From the weight of Z it then follows that $\omega(Q_1(p)) \geq a$. The weight of points below p in S is at least $2b$, therefore, $\omega(Q_4(p)) < b$ implies that $\omega(Q_3(p)) \geq b$. We have thus found an appropriate pair of opposite quadrants. $\square$

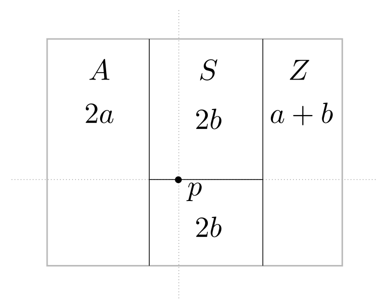

Figure 4: An illustration of the proof of Proposition 2.1.

Setting $a = \frac{1}{3}$ and $b = 0$ in the above proposition implies that $(\frac{1}{3}, 0)$ is admissable. To see that $(\frac{1}{3}+\varepsilon, 0+\varepsilon)$ is not admissible for any $\varepsilon > 0$ it is enough to consider three independent chains each of weight $\frac{1}{3}$ (The top right of Figure 2 depicts four independent chains.). In this example $\omega(Q(p)) > \frac{1}{3}$ implies $\omega(Q^{\mathsf{op}}(p)) = 0$. Therefore:

Observation 2.2. *Pairs (a, b) with $a > \frac{1}{3}$ and $b > 0$ are not admissible, i.e., they are not in $\mathcal{F}$.*

The next observation implies that $(\frac{1}{2}, 0)$ belongs to the boundary of $\mathcal{F}$.

Observation 2.3. *The pair $(\frac{1}{2}, 0)$ belongs to $\mathcal{F}$. Moreover, for every $a > \frac{1}{2}$ the pair $(a, 0)$ does not belong to $\mathcal{F}$.*

Proof. Let P be a set of n points in the plane and let $p \in P$ be the point of P with the largest x_1-coordinate. All points of $P \setminus \{p\}$ are contained in the second and third quadrant of p. Hence, one of these two quadrants has weight at least $\frac{1}{2}$. This shows that $(\frac{1}{2}, 0)$ belongs to $\mathcal{F}$.

To see that $(a, 0)$ does not belong to $\mathcal{F}$ for any $a > \frac{1}{2}$, consider a set P of n points evenly distributed on a circle, or equivalently the vertices of a regular n-gon. It is left to the reader to verify that no point in P has a quadrant of weight greater than $\frac{1}{2}$. $\square$

We now get to the concavity at $(\frac{1}{5}, \frac{1}{10})$ on the boundary of $\mathcal{F}$ and the two segments bounding $\mathcal{F}$ that meet in this point.

Theorem 2.4. *Every pair (a, b) with $2a + 6b = 1$ and $a \leq \frac{1}{4}$ is in $\mathcal{F}$.*

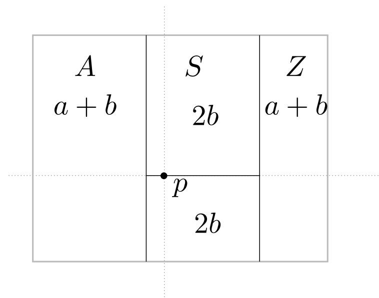

Figure 5: An illustration of the proof of Theorem 2.4.

Proof. Given a set P of points and a pair (a, b), choose two vertical lines, such that the set A of points to the left of both lines has weight $a + b$, the set S in the strip between the lines has weight $4b$ and the set Z of points to the right of the two lines has the remaining weight $a + b$. Consider the horizontal median point p of the middle set S (see Figure 5).

One of the quadrants of p has weight at least $\frac{1}{4}$. Without loss of generality we assume that this is true for $Q_1(p)$. The restriction $\frac{1}{4} \geq a$ implies $\omega(Q_1(p)) \geq a$. If $\omega(Q_3(p)) \geq b$ we are done. Hence, we may assume that $\omega(Q_3(p)) < b$. From the weight of A it then follows that $\omega(Q_2(p)) \geq a$. Similarly from the weight of points below p in S it follows that $\omega(Q_4(p)) \geq b$. Hence, (Q_2, Q_4) is an appropriate pair of opposite quadrants. $\square$

Theorem 2.5. *Every pair (a, b) with $4a + 2b = 1$ and $\frac{3}{16} \leq a$ is in $\mathcal{F}$.*

Proof. In the proof we will need that $2a \geq 3b$, which follows from $4a + 2b = 1$ and $\frac{3}{16} \leq a$.

Given a set P of points and a pair (a, b), choose two vertical lines, such that the set A of points to the left of both lines has weight $2a - b$, the set S in the strip between the lines has weight $4b$ and the set Z of points to the right of the two lines has the remaining weight $2a - b$. Consider the horizontal median point p of the middle set S (see Figure 6).

One of the quadrants of p has weight at least $\frac{1}{4}$. Without loss of generality we assume that this is true for $Q_1(p)$, i.e., $\omega(Q_1(p)) \geq \frac{1}{4} \geq a$. If $\omega(Q_3(p)) \geq b$ we are done. Hence, we may assume that $\omega(Q_3(p)) < b$. From the weight of points below p it then follows that $\omega(Q_4(p)) \geq b$. Since the weight of A is at least $2a - b \geq 2b$ it follows that $\omega(Q_2(p)) \geq b$. If one of the quadrants $Q_2(p)$ and $Q_4(p)$ has weight at least a we are done. But if $\omega(Q_2(p)) < a$ and $\omega(Q_4(p)) < a$ the points in the union of the second, third and fourth quadrant of p would have total weight less than $2a + b$ which contradicts the choice of p. $\square$

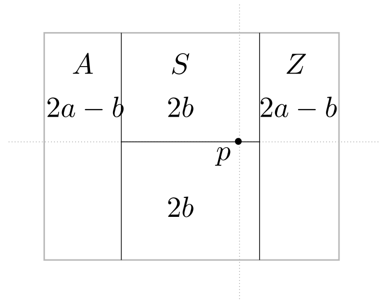

Figure 6: An illustration of the proof of Theorem 2.5.

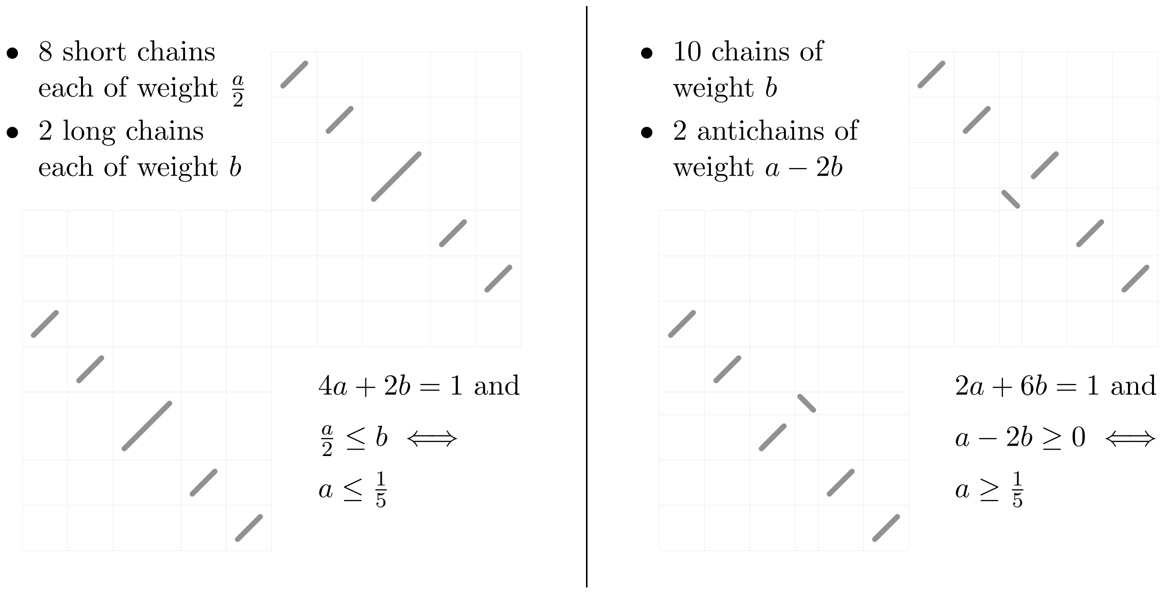

Figure 7: Two examples showing that the pairs (a, b) from Theorem 2.5 and Theorem 2.4 are on the boundary of $\mathcal{F}$. The analysis is again based on a simple case distinction.

It remains to show that for $\frac{1}{4} \le a \le \frac{1}{3}$ the boundary of $\mathcal{F}$ is a segment supported by the line $3a+3b=1$. The following theorem shows that all pairs (a, b) on this segment are admissible. In Figure 9 we present a point set P with no point having opposite quadrants that are $(a+\varepsilon, b+\varepsilon)$-admissible for $3a+3b=1$, $a \ge \frac{1}{4}$, and any $\varepsilon > 0$. The analysis of the example is given in Proposition 2.7.

Theorem 2.6. *Every pair (a, b) with $3a+3b=1$ and $\frac{1}{4} \le a \le \frac{1}{3}$ is in $\mathcal{F}$.*

Proof. Given a set P of points and a pair (a, b), choose a vertical line, such that the set A of points to the left of the line has weight $a+b$. Choose another vertical line, such that the set Z of points to the right of this line has weight $a+b$. The set S in the strip between the lines also has weight $a+b$. This set S is divided vertically into a top part T, a middle part M and a bottom part B, such that each of T and B has weight $2b$ (see Figure 8). The vertical lines are chosen, such that the weight of M is $(a+b)-4b = a-3b$. From $a \ge \frac{1}{4}$

and $3a + 3b = 1$ it follows that this weight is non-negative.

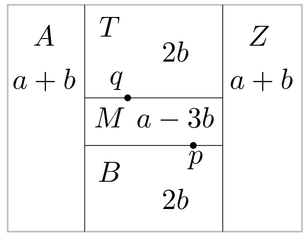

Figure 8: An illustration of the proof of Theorem 2.6.

Now let q be the point on a line separating T and M and p be the point on a line separating M and B, see Fig. 8. Suppose one of the quadrants of p or q has weight at least a. In this case we can simply disregard the middle part M and follow the very same argumentation as in Theorem 2.4 to find an appropriate pair of opposite quadrants.

To see that at least one of the quadrants of p or q has weight at least a we sum up the weights of the first and second quadrant of p and the third and fourth quadrant of q: $\omega(Q_1(p)) + \omega(Q_2(p)) + \omega(Q_3(q)) + \omega(Q_4(q)) \geq 4a$. □

Proposition 2.7. *No pair $(a+\varepsilon, b+\varepsilon)$ with $a \geq \frac{1}{4}$, $\varepsilon > 0$, and $3a + 3b = 1$ is in $\mathcal{F}$.*

Proof. As in the previous proof we will need that $a - 3b \geq 0$, which follows from $a \geq \frac{1}{4}$ and $3a + 3b = 1$.

The example consists of four chains/antichains each of weight $\frac{1}{2}(a - 3b)$, a circle of weight $a - 3b$ and 12 chains/antichains each of weight b (see Figure 9). The total weight of the set S_1 consisting of an antichain and three chains is $\frac{1}{2}(a-3b) + 3b = \frac{1}{2}(a+3b)$. Since $a - 3b \geq 0$ this is at most a.

Since the example is invariant under rotations of 90 degrees it is enough to show that there is no point p, such that $(Q_1(p), Q_3(p))$ is $(a+\varepsilon, b+\varepsilon)$-admissible for $\varepsilon > 0$. Since we need $\omega(Q_1(p)) > a$ we cannot take p from S_1 or S_2. When considering $p \in S_4$ we get strong restrictions from the requirement $\omega(Q_3(p)) > b$. To match this we need to have two of the antichains of size b of S_4 in $Q_3(p)$. Therefore $Q_1(p)$ can only contain the chain of size $\frac{1}{2}(a-3b)$ from S_4, which yields $\omega(Q_1(p)) \leq \frac{1}{2}(a-3b) + \frac{1}{2}(a+3b) = a$, which is not enough.

Any point p from the circle C has $\omega(Q_1(p) \cap C) \leq \frac{1}{2}\omega(C) = \frac{1}{2}(a-3b)$. This rules out points from the circle because for such points $\omega(Q_1(p)) \leq \frac{1}{2}(a-3b) + \frac{1}{2}(a+3b) = a$ which is not enough.

The last possibility is to take $p \in S_3$. Such a point, however, does not match the requirement $\omega(Q_3(p)) > b$.

Altogether this shows that there is no point with an $(a+\varepsilon, b+\varepsilon)$-admissible pair of opposite quadrants. □

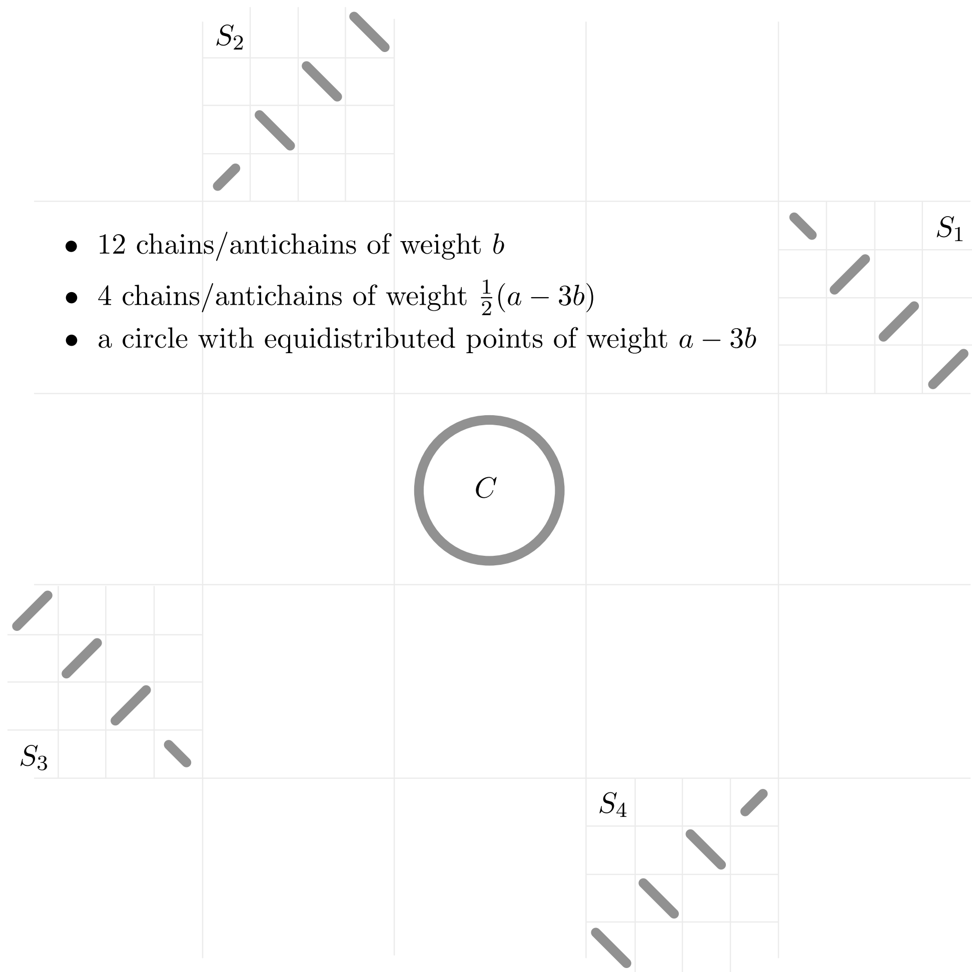

Figure 9: An example showing that there are no admissible pairs $(a+\varepsilon, b+\varepsilon)$ beyond the segment $3a+3b=1$. The analysis is in Proposition 2.7

3 Higher Dimensions

A point p in $\mathbb{R}^d$ defines 2^d orthants centered at p. Again there is an obvious notion of an orthant O^{op} opposite to a given orthant O. The weight $\omega(O)$ of an orthant O with respect to a point set P is the fraction of points of P contained in O. For a more formal definition of the weight we refer to the introduction.

Define the *opposite-orthant depth* α_d for point sets in $\mathbb{R}^d$ as the maximum a, such that every point set $P \subset \mathbb{R}^d$ contains a point p that determines a pair (O, O^{op}) of opposite orthants with $\omega(O) \geq a$ and $\omega(O^{\mathsf{op}}) \geq a$. Brönnimann et al. [BLP] have considered α_3. They claim that $\alpha_3 \geq 1/2016$, this however is based on the false assumption that every set of 9 points in $\mathbb{R}^3$ has a point p with two opposite orthants each containing a point from $P \setminus \{p\}$. Indeed, the least n such that this holds is $n = 17$. This is related to the Erdős-Szekeres lemma, details can be found in the proof of Thm. 3 of [Fel]. With the correct value

17 the proof of [BLP] only yields $\alpha_3 \geq 1/16320$. The case $d = 3$ of the theorem below gives $\alpha_3 \geq 2^{-7} = 1/128$.

For a point $x \in \mathbb{R}^d$ and $i = 1, \ldots, d$ define the closed halfspaces $H_i^+(x) = \{y : y_i \geq x_i\}$ and $H_i^-(x) = \{y : y_i \leq x_i\}$. A *sign vector* is a d-tuple $\sigma = (\sigma_1, \ldots, \sigma_d)$ with $\sigma_i \in \{+, -\}$. For every point x and sign vector σ we define the orthant $O^\sigma(x) = \bigcap_i H_i^{\sigma_i}(x)$.

Theorem 3.1 (Lower Bound)**.** *In $\mathbb{R}^d$, the opposite-orthant depth α_d is at least $2^{-(2^{d-1}+d)}$.*

Proof. A set of points is *t-good* if it contains no point determining a pair of opposite orthants each containing $t + 2$ or more points from P. We will prove that $|P| > 2^{2^{d-1}}(t\,2^d)$ implies that P is not t-good. Hence $\alpha_d(t2^{2^{d-1}+d} + 1) \geq t + 2$, which yields the bound stated in the theorem.

Let P be a t-good set. One of the orthants from each pair of opposite orthants determined by $p \in P$ is *small* in the sense that it contains at most t points from $P \setminus \{p\}$. The *pattern* assigned to p is a collection $\phi(p)$ of 2^{d-1} sign vectors, such that

- $O^\sigma(p)$ is small for each $\sigma \in \phi(p)$ and for each pair $(O^\sigma, O^{\bar{\sigma}})$ of opposite orthants determined by p either σ or $\bar{\sigma}$ is in $\phi(p)$.

For a given pattern ϕ we collect all points $p \in P$ with $\phi(p) = \phi$ in a set P_ϕ. Figure 10 shows an example.

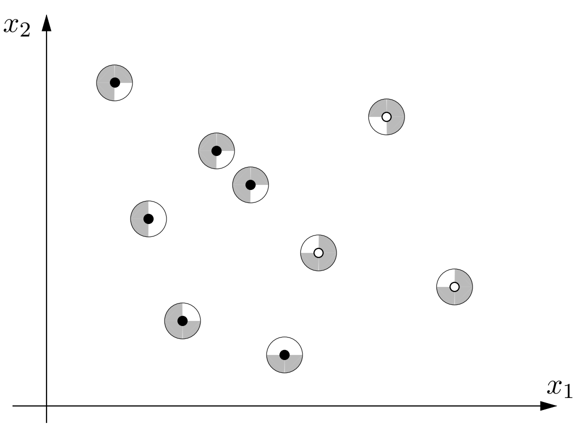

Figure 10: A 2-good set of 9 points in the plane. Small quadrants of points are indicated by gray angles. The white points may get the pattern $\phi = \{(+,+), (+,-)\}$ assigned.

We have partitioned the points of P according to their pattern. The upper bound on the size of any t-good set P follows from counting the possible patterns and bounding the number of points in each class P_ϕ.

There are 2^d sign vectors paired up in 2^{d-1} pairs $\sigma, \bar{\sigma}$ belonging to pairs of opposite orthants. A pattern is a selection of one sign vector from each such pair, therefore:

- There are at most $2^{2^{d-1}}$ different patterns.

For any p and σ let $v_\sigma(p) = |O^\sigma(p) \cap (P \setminus \{p\})|$. Define the *score* of p as $s(p) = \sum_{\sigma \in \phi(p)} v_\sigma(p)$.

From the definition of $\phi(p)$ it follows that $v_\sigma(p) \le t$ for all $\sigma \in \phi(p)$, hence, $s(p) \le t\, 2^{d-1}$. Note that the score of p is the number of points in the small orthants $O^\sigma(p)$ with $\sigma \in \phi(p)$.

Consider two points p, q and note that $q \in O^\sigma(p)$ if and only if $p \in O^{\bar\sigma}(q)$. Suppose p and q both belong to P_ϕ, since one of σ and $\bar\sigma$ is in ϕ we note that either p is counted in the score of q or q is counted in the score of p. From this we obtain:

$$\frac{|P_\phi|(|P_\phi|-1)}{2} = \binom{|P_\phi|}{2} \le \sum_{p\in P_\phi} s(p) \le \sum_{p\in P_\phi} t\, 2^{d-1} = |P_\phi|\, t\, 2^{d-1}.$$

To reduce the upper bound on $\sum s(p)$ by one observe that for each $\sigma \in \phi$ there are points in P_ϕ with $v_\sigma(p) < t$. This yields the following bound on $|P_\phi|$:

- For each class P_ϕ we have $|P_\phi| \le 2\, t\, 2^{d-1}$.

Combining the bounds for the number of patterns and the size of the classes we find that a t-good set P has at most $2^{2^{d-1}}(t\, 2^d)$ points. □

The upper bound on the opposite-orthant depth α_d presented in the following theorem is only a factor of two apart from the lower bound of Theorem 3.1. It is evident that the lower bound is not tight. In dimension 2 the upper and lower bounds yield $\frac{1}{16} \le \alpha_2 \le \frac{1}{8}$. From Theorem 1.1 and the example of Figure 2 we know that $\alpha_2 = \frac{1}{8}$. Indeed we suspect that in all dimensions the upper bound gives the true value of α_d.

Theorem 3.2 (Upper Bound)**.** *In $\mathbb{R}^d$, the opposite-orthant depth α_d is at most $2^{-(2^{d-1}+d-1)}$.*

Proof. We have to construct large point sets with small opposite-orthant depth. The construction is in two steps. In the first step we build a set P_0 of $2^{2^{d-1}}$ points, such that for any point $p \in P_0$ and any pair (O, O^{op}) of opposite orthants at p either $O \cap P_0$ or $O^{\mathsf{op}} \cap P_0$ equals $\{p\}$. In the second step we replace each point of P_0 with a carefully chosen set of $t\, 2^{d-1}$ points, such that the depth remains bounded by $t+1$. Hence $t+2 \ge \alpha_d(2^{2^{d-1}}\, t\, 2^{d-1})$, which yields the bound stated in the theorem.

Let σ be a sign vector and $\bar\sigma$ be the sign vector of the orthant opposite to O^σ. Based on σ we define a binary relation on P, let $p \sim_\sigma q$ if $p \in O^\sigma(q)$ or $p \in O^{\bar\sigma}(q)$, i.e., $q \in O^\sigma(p)$.

A set M of points in $\mathbb{R}^d$ is *monotone* if there is a sign vector σ, such that $p \sim_\sigma q$ for all $p, q \in M$. Equivalently, M is monotone if there is an ordering of the points so that each coordinate is increasing or decreasing in this order. Repeated application of the Erdős-Szekeres lemma implies that any n points in $\mathbb{R}^d$ contain a monotone subset of size at least $n^{\frac{1}{2^{d-1}}}$. This bound is best possible. A detailed construction of tight examples can be found e.g. in [Lit]. Due to this result there is a set P_0 of $2^{2^{d-1}}$ points that does not contain a monotone subset of size three. Hence, for every $p \in P_0$ and every pair of opposite orthants (O, O^{op}) determined by p at least one of the orthants contains no point of $P_0 \setminus \{p\}$.

An orthant $O^\sigma(p)$ defined by $p \in P_0$ is *small* if $O^\sigma(p) \cap P_0 = \{p\}$. As in the proof of the previous theorem we collect sign vectors of small orthants of $p \in P_0$ in a pattern $\phi(p)$. Recall that $\phi(p)$ contains the sign vector of one of each pair of opposite orthants. We construct the set P by replacing each point $p \in P_0$ by a set $Q(p)$ of $t\, 2^{d-1}$ points, such that

$q \in Q(p)$ and $\sigma \in \phi(p)$ implies $|O^\sigma(q) \cap P| = |O^\sigma(q) \cap Q(p)| \le t+1$, i.e., the orthant $O^\sigma(q)$ is t-small for all $\sigma \in \phi(p)$.

For each $p \in P_0$ fix a 'small' box $B(p)$ containing p, such that every choice of one point from each of these boxes yields a set with the same property as P_0. Formally, for $p, q \in P_0$ with $p \sim_\sigma q$ we require that $p' \sim_\sigma q'$ for all $p' \in B(p)$ and $q' \in B(q)$. For the construction of $Q(p)$ in the box $B(p)$ it is convenient to think of $B(p)$ as an open set. To begin with let $\sigma_0, \sigma_1, \sigma_2, \ldots$ be an ordering of the 2^{d-1} sign vectors in $\phi(p)$. Starting with $S_0 = \emptyset$ we inductively define subboxes S_i for $i = 1, \ldots, 2^{d-1} - 1$ of $B_0 = B(p)$ as follows: If S_{i-1} and B_{i-1} are defined choose a point s_i in B_{i-1} and let $S_i = O^{\sigma_i}(s_i) \cap B_{i-1}$ and $B_i = O^{\bar{\sigma}_i}(s_i) \cap B_{i-1}$. Finally, let $S_{2^{d-1}} = B_{2^{d-1}-1}$. An example is given in Figure 11.

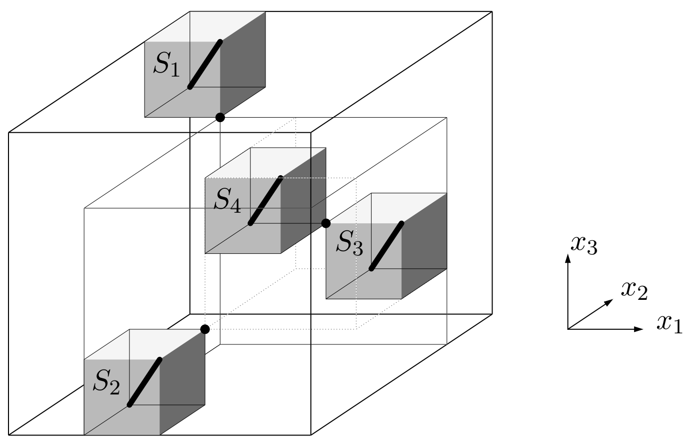

Figure 11: A point set $Q(p)$. The points of $Q(p)$ are aligned along diagonals of the subboxes S_i as indicated by the black bars. The pattern $\phi(p)$ of the point replaced by $Q(p)$ is $\sigma_0 = (-,+,-)$, $\sigma_1 = (-,+,+)$, $\sigma_2 = (-,-,-)$ and $\sigma_3 = (+,+,-)$.

From the construction rules it follows that for $i < j$ and any points $p_i \in S_i$ and $p_j \in S_j$ we have $p_j = O^{\bar{\sigma}_i}(p_i)$ and consequently $p_i = O^{\sigma_i}(p_j)$. This shows that inserting t points into each S_i such that any two of these points are in relation σ_0 yields a set $Q(p)$ with the desired properties.

- $|Q(p)| = t\, 2^{d-1}$
- $|O^\sigma(q) \cap P| = |O^\sigma(q) \cap Q(p)| \le t+1$ for all $q \in Q(p)$ and $\sigma \in \phi(p)$.

This completes the construction of a t-good set P and the proof of the upper bound on α_d. □

4 Further Generalizations

The set $\mathcal{F}$ of all admissible pairs was defined as the set of all pairs (a, b) such that every set P of n points in the plane contains a point $p \in P$ that determines two opposite quadrants with weights at least a and b, respectively. What if we do not require the point p to belong to the set P, and only look for a point z in the plane with the same property of two opposite quadrants determined by it. It is not hard to see that (a, b) is admissible in this model whenever, $a + b \leq 1/2$. In addition we have the admissible pairs $(a, 0)$ for all $a \leq 1$. A set of points uniformly distributed on a circle C shows that this is the complete description of the set of admissible pairs. Indeed, it is not hard to check that for any point z in the plane, surrounded by C, the horizontal and vertical lines through z determine four quadrants such that the measure of the union of any two opposite ones is at exactly $\frac{1}{2}$. If the point z is not surrounded by C, then the measure of one quadrant is equal to 0 while the opposite quadrant has measure at least $\frac{1}{2}$.

One way to generalize the setting is by considering two opposite quadrants determined by a vertical and a horizontal line as a *diagonal* in the 2-by-2 array of cells determined by these two lines. Then it is natural to consider $n-1$ vertical lines and $n-1$ horizontal lines and to find a *generalized diagonal* of cells in the n-by-n arrangement of cells, determined by these lines, such that each of the cells contains 'many' points of P. By a generalized diagonal we mean a set of n cells, such that no two are in the same row or in the same column (see Figure 12).

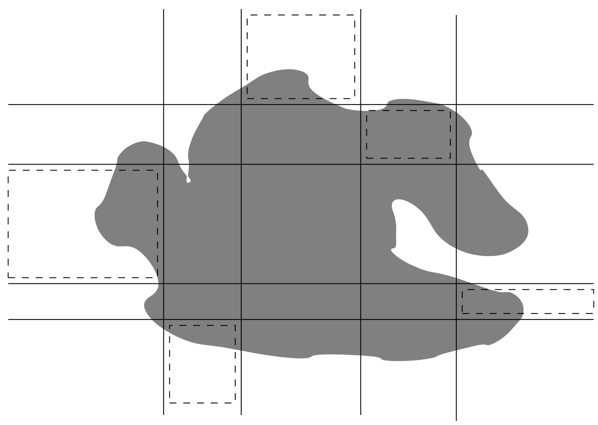

Figure 12: A continuous measure split into cells by four vertical and four horizontal lines. A generalized diagonal is illustrated in the dotted squares.

Since we do not require the vertical and horizontal lines to intersect in points of the set in question, it is equivalent to consider a continuous probability measure in the plane and to look for $n - 1$ vertical lines and $n - 1$ horizontal lines and a generalized diagonal in the arrangement of the n-by-n cells, determined by these lines, such that each of the cells has 'large' measure (see Figure 12). This naturally generalizes to higher dimensions as well. The following theorem gives a partial answer to this problem in any dimension d:

Theorem 4.1. *Let μ be a continuous probability measure in $[0, 1]^d$. Let n be a positive integer and let $\alpha_1, \ldots, \alpha_n$ be a sequence of positive real numbers, such that $\sum_{i=1}^{n} \alpha_i = \frac{1}{n^{d-1}}$. Then there exist numbers $x_{i,j}$, where $1 \leq i \leq d$ and $0 \leq j \leq n$, and d permutations $\pi_1, \ldots, \pi_d$ on $\{1, \ldots, n\}$ with the following properties:*

1. *For every $1 \leq i \leq d$, $x_{i,0} = 0 < x_{i,1} < \ldots < x_{i,n-1} < x_{i,n} = 1$.*

2. *For every* $1 \leq j \leq n$, *we have* $\mu([x_{1,\pi_1(j)-1}, x_{1,\pi_1(j)}] \times \ldots \times [x_{d,\pi_d(j)-1}, x_{d,\pi_d(j)}]) \geq \alpha_j$.

Remark. Intuitively, the numbers $x_{i,j}$ in Theorem 4.1 define the d dimensional array of n^d cells generated by the hyper-planes $H_{i,j} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i = x_{i,j}\}$, for all $1 \leq i \leq n$ and $1 \leq j \leq d$. The theorem then says that there exists such an array (while the measure μ is given) that contains a generalized diagonal of n boxes with measures of at least $\alpha_1, \ldots, \alpha_n$, respectively.

Proof. In fact, the proof is not much more complicated than the statement of the theorem. The proof goes by induction on d. For $d = 1$ simply define $x_{1,0} = 0$ and for every $1 \leq j \leq n$ let $x_{1,j} = x_{1,j-1} + \alpha_j$. In this case $\pi_1 : [n] \to [n]$ can be chosen to be the identity permutation.

For $d > 1$, let $\alpha'_j = n\alpha_j$ for every $1 \leq j \leq n$. Observe that $\sum_{j=1}^{n} \alpha'_j = \frac{1}{n^{d-2}}$. Let μ' denote the measure μ projected along the last dimension. That is, μ' is a continuous probability measure on $[0,1]^{d-1}$ defined by $\mu'([a_1, b_1] \times \ldots \times [a_{d-1}, b_{d-1}]) = \mu([a_1, b_1] \times \ldots \times [a_{d-1}, b_{d-1}] \times [0, 1])$.

By the induction hypothesis there exist numbers $x_{i,j}$, where $1 \leq i \leq d-1$ and $0 \leq j \leq n$, and $d-1$ permutations $\pi_1, \ldots, \pi_{d-1}$ on the elements $\{1, \ldots, n\}$, with the following properties:

1. For every $1 \leq i \leq d-1$ $x_{i,0} = 0 < x_{i,1} < \ldots < x_{i,n-1} < x_{i,n} = 1$.
2. For every $1 \leq j \leq n$ we have:
   $\mu'([x_{1,\pi_1(j)-1}, x_{1,\pi_1(j)}] \times \ldots \times [x_{d-1,\pi_{d-1}(j)-1}, x_{d-1,\pi_{d-1}(j)}]) \geq \alpha'_j$.

We now define the sequence $x_{d,0}, x_{d,1} \ldots, x_{d,n}$ and the permutation π_d as follows. We put $x_{d,0} = 0$. $x_{d,1}$ is defined to be the minimum number greater than $x_{d,0}$, such that there exists some j between 1 and n with $\mu([x_{1,\pi_1(j)-1}, x_{1,\pi_1(j)}] \times \ldots \times [x_{d-1,\pi_{d-1}(j)-1}, x_{d-1,\pi_{d-1}(j)}] \times [x_{d,0}, x_{d,1}]) = \alpha_j$. We set $\pi_d(j) = 1$. From the choice of j and $x_{d,1}$ it follows that for every $1 \leq k \leq n$ we have $\mu([x_{1,\pi_1(k)-1}, x_{1,\pi_1(k)}] \times \ldots \times [x_{d-1,\pi_{d-1}(k)-1}, x_{d-1,\pi_{d-1}(k)}] \times [x_{d,0}, x_{d,1}]) \leq \alpha_k$.

For $1 < k \leq n$ we define $x_{d,k}$ to be the minimum number greater than $x_{d,k-1}$, such that $\mu([x_{1,\pi_1(j)-1}, x_{1,\pi_1(j)}] \times \ldots \times [x_{d-1,\pi_{d-1}(j)-1}, x_{d-1,\pi_{d-1}(j)}] \times [x_{d,k-1}, x_{d,k}]) = \alpha_j$ and there exists some j between 1 and n, such that j is different from each of $\pi_d^{-1}(1), \ldots, \pi_d^{-1}(k-1)$. We set $\pi_d(j) = k$.

Finally, we let $x_{d,n} = 1$ and set $\pi_d(j) = n$, where j is the only index between 1 and n that is not one of $\pi_d^{-1}(1), \ldots, \pi_d^{-1}(n-1)$.

It is straight forward to check that the numbers $x_{i,j}$ where $1 \leq i \leq d$ and $0 \leq j \leq n$ and $\pi_1, \ldots, \pi_d$ satisfy the requirements of the theorem. □

Recall that when $d = 2$ and $n = 2$, the example where the measure μ is evenly distributed on a circle C shows that the result in Theorem 4.1 is best possible apart from the case where $\alpha_i = 0$ for some i. When the values α_i in Theorem 4.1 are all equal, then the result is best possible in any dimension. To see this just consider the standard Lebesgue measure on $[0,1]^d$. It is then not very difficult to verify the following proposition:

Proposition 4.2. *Let λ be the standard Lebesgue measure on $[0,1]^d$ and let n be a positive integer. Then one cannot find numbers $x_{i,j}$, where $1 \le i \le d$ and $0 \le j \le n$, such that the arrangement of n^d bounded cells determined by the hyperplanes $H_{i,j} = \{z : z_i = x_{i,j}\}$ has a generalized diagonal, such that each cell of the generalized diagonal has measure strictly larger than $1/n^d$.*

Below we give some hints for the proof but leave the details to the reader. Before that let us mention that even in the case $d = 3$ and $n = 2$ the result in Theorem 4.1 is not best possible for general values of α_1 and α_2.

One can show the following statement is true. Let P be a finite set of points in $\mathbb{R}^3$. We assume that no two points in P have a coordinate in common. Let $\alpha_1 \le \frac{1}{12}$ and $\alpha_2 = \frac{1}{3} - 3\alpha_1$. Then there exists a point $z = (z_1, z_2, z_3) \in \mathbb{R}^3$, such that the planes $\{x_i = z_i\}$ for $i = 1, 2, 3$ determine two opposite orthants, one with at least $\alpha_1 n$ points and the other with at least $\alpha_2 n$ points.

Surprisingly, this statement is nearly tight when α_1 is close to 0 (and thus α_2 is close to $\frac{1}{3}$). To see this consider the following construction of a set $P \subset \mathbb{R}^3$ of $n = 3m$ points. It has the property that for any point z and any two opposite orthants, determined by z, such that each contains at least one point of P, there are at most $m = \frac{1}{3}n$ points of P in each of these orthants.

To construct P let S be the following set of m vectors in $\mathbb{R}^3$, very close to the origin: $S = \{\frac{i}{100m}(1,1,1) \mid i = 0, 1, \ldots, m-1\}$. We define A, B, and C to be the set S translated by $(-1,0,1)$, $(0,1,-1)$, and $(1,-1,0)$ respectively. Finally we let $P = A \cup B \cup C$. Note that P is a set of $n = 3m$ points in $\mathbb{R}^3$ no two of which share the same x_1, x_2, or x_3 coordinates.

We leave it to the reader to verify that P satisfies the required properties. We also leave the challenge of giving a full characterization of such admissible pairs α_1 and α_2.

5 Conclusion

We gave a complete description of the set $\mathcal{F}$ of all admissible pairs (a, b) in the plane. This was done by identifying three line segments and the points $(\frac{1}{8}, \frac{1}{8})$ and $(0, \frac{1}{2})$ from the boundary of $\mathcal{F}$.

In higher dimensions we were interested in the maximum number α_d, such that (α_d, α_d) is admissible. We think that our upper bound on α_d is tight, the lower bound leaves room for improvements. It would be interesting to get more information about the set $\mathcal{F}_d$ of all admissible pairs (a, b) for $\mathbb{R}^d$.

In the relaxed setting, where the point determining a pair of opposite orthants need not belong to the point set we could determine the diagonal entry precisely, it is $(\frac{1}{2^d}, \frac{1}{2^d})$. This result follows from more general bounds on generalized diagonals. In this relaxed setting we have indicated some additional bounds for admissible pairs (a, b) but many questions remain.

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