AN EXPLICIT PL-EMBEDDING OF THE FLAT SQUARE TORUS INTO $\mathbb{E}^3$

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Abstract. We present an explicit PL-embedding of the flat square torus $T^2 = \mathbb{E}^2 / \mathbb{Z}^2$ into $\mathbb{E}^3$, with 40 vertices and 80 faces.

1 Introduction

In [4], Burago and Zalgaller proved that any connected compact polyhedral surface admitting a topological embedding, admits an isometric piecewise linear (PL) embedding into $\mathbb{E}^3$. Recall that a polyhedral surface is a 2-dimensional manifold endowed with a polyhedral metric, i.e., a metric such that every point has a neighborhood isometric to the neighborhood of the vertex of a cone in $\mathbb{E}^3$. A PL map is defined as follows:

Definition 1. Let $S$ be a polyhedral surface. A map $f : S \to \mathbb{E}^3$ is said piecewise linear (PL) if $S$ admits a triangulation such that the restriction of $f$ to any triangle is linear, i.e., it preserves barycentric coordinates. The PL map $f$ is piecewise distance preserving if $S$ admits a triangulation such that the restriction to any triangle is distance preserving, i.e., $d_{\mathbb{E}^3}(f(x), f(y)) = d_S(x, y)$ for any $x, y$ in a same triangle.

Note that once the triangulation $T$ is given, the images of the vertices determine a unique PL map by extending linearly to the images of triangles of $T$ in $\mathbb{E}^3$. A PL map $f : S \to \mathbb{E}^3$ is an embedding if it is an homeomorphism between $S$ and $f(S)$. If $S$ is compact, a PL embedding is a PL injective map.

The approach of Burago and Zalgaller’s result relies on the Nash-Kuiper $C^1$-embedding Theorem ([7, 6]) and their construction is not explicit for this reason (see [9] for a discussion). In addition to an initial PL-approximation of an almost $C^1$ isometric embedding, the construction of Burago-Zalgaller involves several subdivision steps so that the resulting triangulation is very large. Finding an explicit triangulation with few vertices appears a real challenge. Zalgaller investigated the question of how to construct explicit PL-embeddings of cylinders or flat tori and found a solution for long cylinders and long tori [10]. Recall that a flat torus is the quotient of the two-dimensional Euclidean plane by a lattice. It is called rectangular when the lattice itself is rectangular. The above construction of Zalgaller restricted to rectangular tori requires that the width is at least twice of its height. In this

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Theorem 1. For any $\lambda \geq 1$ consider a rectangle $ABCD$ of sides 1 and $\lambda$. Denote by $T_\lambda$ the polyhedral surface obtained by identifying the opposite sides $\overline{AB}$ and $\overline{DC}$, $\overline{AD}$ and $\overline{BC}$. Then, there is a triangulation of $T_\lambda$ with 40 vertices admitting an (PL) isometric embedding into $\mathbb{R}^3$ which is linear on each face of the triangulation (See Figure 1).

Our PL-embedding is inspired by the corrugated $C^1$ isometric embeddings of the flat torus generated by the Convex Integration Theory and constructed in [1, 2]. Essentially, we construct PL corrugations along one side of the rectangle to introduce flexibility and to allow the identifications between the opposite sides. We show that five corrugations are enough to obtain a PL-isometric embedding. The corresponding number of vertices is 40. If the isometric constraint is dropped, it is known that the torus admits a PL-embedding with 7 vertices and that this number can not be reduced [5, 3]. The question of the minimum number of vertices of a PL isometric embedding of a flat torus is quite natural but probably very difficult.

In the next three sections we focus on the case $\lambda = 1$, that is the one of the square torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. In section 2 we built a triangulation $\mathcal{T}$ of $T^2$ with 40 vertices. In section 3 we describe a linear embedding of $\mathcal{T}$ into $\mathbb{R}^3$. This linear embedding induces a PL map $F : T^2 \to \mathbb{R}^3$. In section 4, we show that the map $F$ is isometric. This proves Theorem 1 for $\lambda = 1$. Section 5 is devoted to the generalisation of this proof to the $\lambda > 1$ case.

2 Triangulations of the square torus

In this section we describe a triangulation $\mathcal{T}$ of the square torus. Let $\mathcal{D} := [0,1]^2$ be a fundamental domain of $\mathbb{R}^2/\mathbb{Z}^2$; Consider the following points in $\mathcal{D}$:
We now locate the points $a'_0$, $b'_0$, $c'_0$ and $d'_0$ by reflecting $a_0$, $b_0$, $c_0$ and $d_0$ through the line $y = \frac{1}{2}$, so that:

\[
\begin{align*}
    a'_0 &= \left(0, \frac{3 - 0.42\ell}{4}\right), & b'_0 &= \left(0, \frac{3 + 3.58\ell}{4}\right), \\
    c'_0 &= \left(\frac{1}{10}, \frac{3 - 0.02\ell}{4}\right), & d'_0 &= \left(\frac{1}{10}, \frac{3 + 0.02\ell}{4}\right)
\end{align*}
\]

Where $\ell$ is given by:

\[
\ell = \frac{\sqrt{0.3759 \sin \frac{\pi}{5}}}{10\sqrt{(0.189 - 0.11\sqrt{0.3759 \cos \frac{\pi}{5}})^2 + 0.003759 \sin^2 \frac{\pi}{5}}} \tag{1}
\]

For $i \in \{1, \ldots, 4\}$ we set $a_i := a_0 + \frac{i}{5}e_1$, ..., $d'_i = d'_0 + \frac{i}{5}e_1$ with $e_1 = (1, 0)$.

To generate a triangulation we start from the trapezoid $a_0b_0c_0d_0$ split along its diagonal $a_0d_0$. We add its reflexion through the line $x = \frac{1}{5}$. We also add the reflexion of these two trapezoids through the line $y = \frac{1}{2}$, obtaining trapezoids $a_i b_i c_i d_i$, $a_{i+1} b_{i+1} c_{i+1} d_{i+1}$ and $a'_i b'_i c'_i d'_i$, $a'_{i+1} b'_{i+1} c'_{i+1} d'_{i+1}$. We finally connect those trapezoids with vertical edges $a_i a'_i$, $b_i b'_i$, $c_i c'_i$ and $d_i d'_i$ together with the diagonals $a_i c'_i$, $a_{i+1} c'_{i+1}$, $b_i d'_i$ and $b_{i+1} d'_{i+1}$ (See Figure 2).

The triangulation $\mathcal{T}$ is built from a pattern which consists of 8 triangles located on a vertical ribbon of width $\frac{1}{10}$ while the other triangles are obtained by reflexion or translation of these 8 triangles (See figure 2).

Each ribbon will be mapped into $\mathbb{E}^3$ to generate half of a PL corrugation (see Figure 3). Preserving the rotational symmetry of the embedding combined with the isometry constraints the geometry of the trapezoids $a_i b_i c_i d_i$. In fact, experimentally moving the points $a_i$, $b_i$, $c_i$ and $d_i$ easily breaks the isometry.
Figure 2: Triangulation $\mathcal{T}$ of the fundamental domain $\mathcal{D}$ and a vertical ribbon $[0, \frac{1}{10}] \times [0, 1]$.

3 PL embeddings of the flat torus

In this section we describe a linear embedding of the triangulation $\mathcal{T}$ into $\mathbb{E}^3$. We denote by $O$ the origin of $\mathbb{E}^3$ and we introduce the three following points of $\mathbb{E}^3$:

\[
\begin{align*}
\Omega_A &= \left(0, 0, \frac{1 - 0.42\ell}{4}\right) \\
\Omega_B &= \left(0, 0, \frac{1 - 3.58\ell}{4}\right) \\
\Omega_* &= \left(0, 0, \frac{1 - 0.02\ell}{4}\right)
\end{align*}
\]  
(2)

We set $v(\theta) = (\cos \theta, \sin \theta, 0)$. We define a PL map $F : \mathbb{T}^2 \to \mathbb{E}^3$ by its image on every vertex of $\mathcal{T}$ and extending by linearity on each face of $\mathcal{T}$:

\[
\begin{align*}
F(a_i) &= \Omega_A + r_A v\left(\frac{2i\pi}{5}\right) \\
F(b_i) &= \Omega_B + r_1 v\left(\frac{2i\pi}{5}\right) \\
F(c_i) &= \Omega_* + r_3 v\left(\frac{(2i+1)\pi}{5}\right) \\
F(d_i) &= \Omega_* + r_2 v\left(\frac{(2i+1)\pi}{5}\right)
\end{align*}
\]  
\]  
(3)

\[
\begin{align*}
F(a_i') &= -\Omega_A + r_A v\left(\frac{2i\pi}{5}\right) \\
F(b_i') &= -\Omega_B + r_1 v\left(\frac{2i\pi}{5}\right) \\
F(c_i') &= -\Omega_* + r_3 v\left(\frac{(2i+1)\pi}{5}\right) \\
F(d_i') &= -\Omega_* + r_2 v\left(\frac{(2i+1)\pi}{5}\right)
\end{align*}
\]  
(3)
Figure 3: Left, view of a slice of $F(T^2)$, on the vertical axis $z_A$, $z_B$ and $z_*$ denote the vertical coordinates of $\Omega_A$, $\Omega_B$ and $\Omega_*$. Right, view from above of $F(T^2)$, the circles have radii $r_1 < r_2 < r_3 < r_4$.

for all $i \in \{0, \ldots, 4\}$ and where $r_1, r_2, r_3$ and $r_4$ are given by:

$$r_4 = \frac{\sqrt{1-\ell^2}}{10 \sin \frac{\pi}{5}}$$
$$r_3 = r_4 \cos \frac{\pi}{5} - 0.1\ell$$
$$r_2 = r_3 - c_0d_0 = r_3 - 0.01\ell$$
$$r_1 = r_4 - \sqrt{0.3759}\ell.$$

Note that the points $F(a_i)$, $i \in \{0, \ldots, 4\}$, lie in a circle of radius $r_4$ and of center $\Omega_A$. Similarly, the points $F(b_i)$, $i \in \{0, \ldots, 4\}$, lie in a circle of radius $r_1$ and of center $\Omega_B$, and so on. Note that $r_1 < r_2 < r_3 < r_4$ (See Figure 3).

**Symmetry property of $F$:** The image torus $F(T^2)$ has some revolution and reflection symmetries. Indeed, we define, for $i = 0, \ldots, 4$, the planes $\Pi_i$ as follows:

$$\Pi_i : x \sin \left(\frac{i\pi}{5}\right) - y \cos \left(\frac{i\pi}{5}\right) = 0.$$  

Let $R_j$ be the rotation about the $z$-axis of angle $\frac{2j\pi}{5}$, and let $S_j$ be the reflection through the plane $\Pi_j$. We claim that $F(T^2)$ is invariant under $S_j$ and $R_j$ for $j = 0, \ldots, 4$.

**Proof of the claim:** Let $v \in \{a, b, c, d, a', b', c', d'\}$, $p \in \{a, b, a', b'\}$ and $q = \{c, d, c', d'\}$. We remark that the list of triangles in $T$ is invariant by the re-indexation $r_j : v_i \to v_{i+j}$.
and $s_j : p_i \rightarrow p_{j-i+5}$ and $s_j : q_i \rightarrow q_{j-i+4}$. From equation (3) we directly have that $R_j(F(v_i)) = F(v_{j+i}) = F(r_j(v_i))$. Moreover, we have $S_j(F(p_i)) = F(p_{j-i+5}) = F(s_j(p_i))$ and $S_j(F(q_i)) = F(q_{j-i+4}) = F(s_j(q_i))$. □

4 Proof of Theorem 1 for $\lambda = 1$

In this section, we prove that the PL map $F : T^2 \rightarrow \mathbb{R}^3$ described by equations (3) in Section 3 is an isometric embedding. To show that $F$ is isometric, it is enough to prove that every triangle of $T$ is mapped isometrically by $F$. In turn, this reduces to show that $F$ preserves the lengths of the edges of the triangulation.

**Proposition 1.** The PL map $F : T^2 \rightarrow \mathbb{R}^3$ defined in equations (3) is isometric.

**Proof.** It is enough to prove that the length of every edge $[p, q]$ in $T$ is preserved under $F$, i.e. $d_{T^2}(F(p), F(q)) = d_{\mathbb{R}^2}(p, q)$. To save space, we often write $pq$ for $d_{T^2}(p, q)$ in this proof.

By the symmetry property of $T$ and $F$, it is thus enough to prove that every edge in the first ribbon of $T$ is preserved by $F$, that is $d_{T^2}(F(p_0), F(q_0)) = p_0q_0$ for all edge $[p_0, q_0]$ in $T$. Direct computations show that:

$$d_{T^2}^2(F(c_0), F(c_0′)) = \left(1 - \frac{0.02\ell}{2}\right)^2 = (c_0c_0′)^2$$

$$d_{T^2}^2(F(a_0), F(a_0′)) = \left(1 - \frac{0.42\ell}{2}\right)^2 = (a_0a_0′)^2$$

$$d_{T^2}^2(F(c_0), F(d_0)) = d_{T^2}^2(F(c_0′), F(d_0′)) = (r_3 - r_2)^2 = (c_0d_0′)^2 = (c_0d_0)^2$$

$$d_{T^2}^2(F(a_0), F(b_0)) = d_{T^2}^2(F(a_0′), F(b_0′)) = (r_4 - r_1)^2 + (0.79\ell)^2 = (a_0b_0′)^2$$

$$d_{T^2}^2(F(a_0), F(c_0)) = d_{T^2}^2(F(a_0′), F(c_0′)) = r_4^2 + r_3^2 - 2r_1r_3\cos\frac{\pi}{5} + (0.1\ell)^2$$

$$= (0.1)^2 + (0.1\ell)^2 = (a_0c_0′)^2 = (a_0c_0)^2$$

$$d_{T^2}^2(F(a_0), F(d_0)) = d_{T^2}^2(F(a_0′), F(d_0′)) = r_4^2 + r_2^2 - 2r_4r_2\cos\frac{\pi}{5} + (0.1\ell)^2$$

$$= (0.1)^2 + (0.11\ell)^2 = (a_0d_0′)^2 = (a_0d_0)^2$$

$$d_{T^2}^2(F(a_0), F(c_0′)) = r_3^2 + r_2^2 - 2r_4r_3\cos\frac{\pi}{5} + \left(1 - \frac{0.22\ell}{2}\right)^2$$

$$= (0.1)^2 + \left(1 - \frac{0.22\ell}{2}\right)^2 = (a_0c_0′)^2$$

$$d_{T^2}(F(b_0), F(b_0′)) = \left(1 - \frac{3.58\ell}{2}\right)^2 = \left(1 - \frac{1 + 3.58\ell}{2}\right)^2 = (b_0b_0′)^2.$$
To do so, we define angular sectors $P_j$ for $j = 0, \ldots, 9$ using the 5 planes $\Pi_i$ defined in equation (5). Let $\Gamma_0$ be the half-plane of $\Pi_0$ such that $x \geq 0$. We denote by $\Gamma_i$ the vertical half-plane bounded by the z-axis and making an angle $\frac{i\pi}{9}$ with $\Gamma_0$ for $i=0, \ldots, 9$. We define

\[ d_{E_3}(F(d_0), F(d_0')) = \left( \frac{1-0.02\ell}{2} \right)^2 = \left( 1 - \frac{1+0.02\ell}{2} \right)^2 = (d_0d_0')^2 \]

\[ d_{E_3}(F(b_0), F(d_0')) = r_1^2 + 2r_2^2 - r_1r_2 \cos \frac{\pi}{5} + \left( \frac{1-1.8\ell}{2} \right)^2 \]

\[ = (0.1)^2 + \left( 1-1.8\ell \right)^2 = (b_0d_0')^2 \]

Finally, we have

\[ d_{E_3}(F(b_0), F(d_0)) = d_{E_3}(F(b_0), F(d_0')) = r_1^2 + r_2^2 - 2r_1r_2 \cos \frac{\pi}{5} + (0.89\ell)^2 \]

\[ = (0.1)^2 + (0.89\ell)^2 + 2 \left[ 0.189\ell^2 - \sqrt{0.3759\ell} \left( r_1 - r_2 \cos \frac{\pi}{5} \right) \right] . \]

The value of $\ell$ given in (1) is such that $0.189\ell^2 - \sqrt{0.3759\ell} \left( r_1 - r_2 \cos \frac{\pi}{5} \right) = 0$, thus we have

\[ d_{E_3}(F(b_0), F(d_0)) = (0.1)^2 + (0.89\ell)^2 = (b_0d_0')^2 = (b_0d_0)^2 \]

as desired. \hfill \Box

**Proposition 2.** The PL map $F : T^2 \to E^3$ is an embedding.

**Proof.** In this proof we often denote by $A$ the point $F(a)$, by $AB$ the segment $F(a)F(b)$, and by $ABC$ the triangle $F(a)F(b)F(c)$ . Let $\mathcal{F}$ be the set of triangles contained in $E^3$ which are images of triangles in $\mathcal{T}$. This set contains 16 families of triangles given below $(i=0, \ldots, 4)$:

1. $A_iB_iD_i$
2. $A_iD_iC_i$
3. $B_iD_iD_i'$
4. $B_iB_i'D_i'$
5. $A_iC_iC_i'$
6. $A_i'C_i'A_i$
7. $A_i'B_i'D_i'$
8. $A_i'D_i'C_i'$
9. $C_iA_{i+1}D_i$
10. $D_iB_{i+1}A_{i+1}$
11. $C_i'A_{i+1}D_i'$
12. $D_i'A_{i+1}B_{i+1}'$
13. $D_iB_{i+1}D_i'$
14. $B_{i+1}D_i'B_{i+1}'$
15. $C_i'C_iA_{i+1}$
16. $C_i'A_{i+1}A_{i+1}$

It is readily checked that $F$ is injective on the vertices of $\mathcal{T}$. Therefore, points $A_i, B_i, \ldots, D_i'$ are all distinct. Moreover, it is straightforward to see that if $T \in \mathcal{F}$, then the vertices of $T$ are not collinear.

The proof reduces to show that for every pair of triangles $T_1, T_2$, such that $T_1 = F(t_1)$ and $T_2 = F(t_2)$ are the images of distinct triangles $t_1$ and $t_2$ in $\mathcal{T}$, we have $T_1 \cap T_2 = F(t_1 \cap t_2)$, i.e., $T_1 \cap T_2$ is either the empty set or a common vertex or a common edge.

To do so, we define angular sectors $P_j$ for $j = 0, \ldots, 9$ using the 5 planes $\Pi_i$ defined in equation (5). Let $\Gamma_0$ be the half-plane of $\Pi_0$ such that $x \geq 0$. We denote by $\Gamma_i$ the vertical half-plane bounded by the z-axis and making an angle $\frac{i\pi}{9}$ with $\Gamma_0$ for $i = 0, \ldots, 9$. We define
Figure 4: View from above of half-planes $\Gamma_i$ and the sectors $\mathcal{P}_j$.

$\mathcal{P}_i$ as the sector of angle $\frac{i\pi}{5}$ delimited by $\Gamma_i$ and $\Gamma_{i+1}$ for $i = 0, \ldots, 8$ and $\mathcal{P}_9$ the sector between $\Gamma_9$ and $\Gamma_0$ with angle $\frac{\pi}{5}$ (See figure 4).

The proof will consist in the following steps:

I. We show that for all $T \in \mathcal{F}$, $\hat{T} \subset \mathcal{P}_j$ for a unique $j = 0, \ldots, 9$.

II. We show that $\mathcal{P}_j$, for $j = 0, \ldots, 9$ contains exactly 8 triangles.

III. We show that for all 28 pairs of triangles $(T_1, T_2)$ in $\mathcal{P}_0$ with $T_1 = F(t_1)$, $T_2 = F(t_2)$, we have that $T_1 \cap T_2 = F(t_1 \cap t_2)$. Due to the symmetry properties of $\mathcal{T}$ and $F$, this result will prove that $F$ is an embedding.

I.

Remark that the angular sectors $\mathcal{P}_i$ are disjoint and that $\mathcal{P}_i \cap \mathcal{P}_{i+1} = \Gamma_{i+1}$. The half-planes $\Gamma_{2i}$ and $\Gamma_{2i+1}$ contain respectively $A_i$, $B_i$, $A'_i$ and $B'_i$ and $C_i$, $D_i$, $C'_i$ and $D'_i$. Since in the list of triangles in $\mathcal{F}$ there are no triangles whose vertices are all contained in a single $\Gamma_i$, there is no triangle contained in $\mathcal{P}_i \cap \mathcal{P}_{i+1}$. Note that the endpoints of each edge of $F(\mathbb{T}^2)$ either have the same index $i$, or have consecutive indexes $i$ and $i+1$. It follows that each edge is contained in some $\mathcal{P}_i$ therefore each triangle interior is contained in some $\mathcal{P}_i$.

II.
Since there are 80 triangles in $\mathcal{T}$ and 10 angular sectors $\mathcal{P}_i$, by symmetry we conclude that each $\mathcal{P}_i$ contains exactly 8 triangles. Then for the symmetry properties of $\mathcal{T}$ and $\mathcal{F}$ we have that it is enough to show that $\mathcal{T}_1 \cap \mathcal{T}_2 = F(t_1 \cap t_2)$ for every pair of triangles $\mathcal{T}_1$ and $\mathcal{T}_2$ contained in $\mathcal{P}_0$. We note that the following 8 triangles are contained in $\mathcal{P}_0$.

1. $A_0B_0D_0$
2. $A_0D_0C_0$
3. $B_0D_0D'_0$
4. $B_0B'_0D'_0$
5. $A_0C_0C'_0$
6. $A'_0C'_0A_0$
7. $A'_0B'_0D'_0$
8. $A'_0D'_0C'_0$

III.

Since we have 8 triangles, we have 28 pairs of triangles to verify, now, since there is also a (combinatorial) symmetry through the plane $z = 0$, we reduce to 6 triangles and then we have only 15 pairs to verify. We denote by $ABC$ the triangle $A_0B_0C_0$. We use the enumeration of the list of triangles for a triangle in $\mathcal{F}$. The 6 triangles to verify are triangles (1)-(6)

**Pairs (1)-(4), (2)-(4):**

Take the plane $P_1$ whose equation is

\[ P_1 : z = \frac{1 - 3.58\ell}{4}. \]

The third coordinate of $B'$ and $D'$ is negative (and thus lesser than $\frac{1-3.58\ell}{4}$) and the third coordinate of $A$, $B$, $C$ and $D$ are greater than $\frac{1-3.58\ell}{4}$ (or equal for the case $B$), we have then that triangles $ABD$ and $ADC$ are in one side of the plane $P_1$ and the triangle $BB'D'$ is in the other side of $P_1$. This shows that $ABD \cap BB'D' = B = F(abd \cap bb'd')$ and $ACD \cap BB'D' = \emptyset = F(acd \cap bb'd')$ as required.

**Pair (2)-(6):**
Take the plane $P_2$ with equation

\[ P_2 : z = \frac{1 - 0.42\ell}{4}. \]

Since the third coordinate of $A$, $C$ and $D$ is greater or equal than $\frac{1-0.42\ell}{4}$ and the third coordinate of $A$, $A'$ and $C'$ is lesser or equal than $\frac{1-0.42\ell}{4}$, the triangles $ACD$ and $A'C'A$ are on opposite sides of the plane $P_2$, so $ACD \cap A'C'A = A = F(acd \cap a'c'a)$ as required.

**Pairs (3)-(5), (4)-(5), (3)-(6) and (4)-(6):**
Let $M$ be the medium point of $CD$ and $M'$ the medium point of $C'D'$. Take the plane $P_3$ passing through $M$ and $M'$ and such that the segment $CD$ is normal to $P_3$. The equation of $P_3$ is given by

\[ P_3 : x \cos \frac{\pi}{5} + y \sin \frac{\pi}{5} - \frac{r_2 + r_3}{2} = 0. \]

We have that $A$ and $C$ are on one side of $P_3$ and $B$ and $D$ on the opposite side of $P_3$. Since the first coordinate of $A'$, $B'$, $C'$ and $D'$ is equal as the first coordinate of $A$, $B$, $C$ and $D$
respectively, we have that triangles $BD'D$ and $B'D'B$ are on one side of $P_3$ and the triangles $ACC'$ and $A'C'A$ are on the other side of $P_3$. Then, $ACC' \cap BD'D = ACC' \cap B'D'B = A'C'A \cap BD'D = A'C'A \cap B'D'B = \emptyset = F(acc' \cap bd'd) = F(acc' \cap b'd'b) = F(a'c'a \cap bd'd) = F(a'c'a \cap b'd'b)$ as desired.

**Pairs (1)-(5), (1)-(6):**
Let $P_4$ be the plane passing through the points $A$, $A'$ and $M$. The equation of $P_4$ is given by

$$P_4 : x \sin \frac{\pi}{5} + y \left(\frac{2r_4}{r_3 + r_2} - \cos \frac{\pi}{5}\right) - r_4 \sin \frac{\pi}{5} = 0.$$  

The coordinates of $D$ and $B$ verify $x \sin \frac{\pi}{5} + y \left(\frac{2r_4}{r_3 + r_2} - \cos \frac{\pi}{5}\right) - r_4 \sin \frac{\pi}{5} < 0$. The coordinates of $C$ and $C'$ verify $x \sin \frac{\pi}{5} + y \left(\frac{2r_4}{r_3 + r_2} - \cos \frac{\pi}{5}\right) - r_4 \sin \frac{\pi}{5} > 0$. This shows that $ADB$ is on one side of $P_4$ and $A'C'A$ and $ACC'$ are on the other side of $P_4$. That means that $ABD \cap A'C'A = ABD \cap ACC' = A = F(abd \cap a'c'a) = F(abd \cap acc').$

**Pair (2)-(3):**
Take the plane $P_5$ passing by $D$ and $D'$ and normal to $CD$. The equation of $P_5$ is given by

$$P_5 : x \cos \frac{\pi}{5} + y \sin \frac{\pi}{5} - r_2 = 0.$$  

We have then that the triangle $ACD$ is on one side of $P_5$ and $BD'D$ is on the other side of $P_5$, this implies that $ACD \cap BD'D = D = F(acd \cap bd'd)$.

**Pairs (1)-(3), (3)-(4), (5)-(6):**
If $P_6$ is the plane passing through $B$, $B'$ and $D$ (and $D'$), the equation describing $P_6$ is given by

$$P_6 : x \sin \frac{\pi}{5} + y \left(\frac{r_1}{r_2} - \cos \frac{\pi}{5}\right) - r_1 \sin \frac{\pi}{5} = 0.$$  

The coordinates of $A$ satisfy $x \sin \frac{\pi}{5} + y \left(\frac{r_1}{r_2} - \cos \frac{\pi}{5}\right) - r_1 \sin \frac{\pi}{5} > 0$. That means that $BD'D$ and $ABD$ are in different planes and since they share the edge $BD$ we have that $BD'D \cap ABD = BD = F(bdl' \cap abd)$. The triangles $BD'D$ and $B'D'B$ are on the plane $P_6$. Since $D$ and $B'$ are on opposite sides of the line of $P_6$ containing the segment $BD'$, we conclude that $BD'D \cap B'D'B = BD' = F(bdl' \cap b'd'b)$. Observe that $A$, $A'$, $C$ and $C'$ are coplanar and so are $ACC'$ and $A'C'A$. In their common plane, $A'$ and $C$ lie on opposite sides of the line containing the segment $AC'$ we have that $ACC' \cap A'C'A = AC' = F(acc' \cap a'c'a)$.

**Pairs (1)-(2), (2)-(5):**
Take the plane $P_7$ passing by $A$, $D$ and $C$. The equation of this plane is given by:

$$P_7 : x 0.1 \ell \sin \frac{\pi}{5} - y 0.1 \ell \cos \frac{\pi}{5} + z r_4 \sin \frac{\pi}{5} - \frac{r_4(1 - 0.02\ell)}{4} \sin \frac{\pi}{5} = 0.$$
\[ B \text{ and } C' \text{ verify that } x \cdot 0.1 \ell \sin \frac{\pi}{5} - y \cdot 0.1 \ell \cos \frac{\pi}{5} + z \cdot r_4 \sin \frac{\pi}{5} - \frac{r_4(1 - 0.02\ell)}{4} \sin \frac{\pi}{5} > 0 \text{ so } ACD \text{ and } ABD \text{ are in different planes. Since they share } AD \text{ the triangles only intersect in that edge. The same happens with the triangles } ACD \text{ and } ACC'. \text{ We have proved then that } ACD \cap ABD = AD = F(acd \cap abd) \text{ and that } ACD \cap ACC' = AC = F(acd \cap acc'). \]

We conclude that \( T_1 \cap T_2 = F(t_1 \cap t_2) \) for every pair of triangles \( T_1 \) and \( T_2 \) contained in \( F \).

\section{Proof of Theorem 1 for \( \lambda > 1 \)}

This proof is an adaptation of the proof for the case \( \lambda = 1 \). In fact, elongating \( F(\mathbb{T}^2) \) vertically in \( \mathbb{E}^3 \) will change the internal angles of the triangles. As a result, the total angles at the vertices will differ from \( 2\pi \) so that the elongated surface will not be flat anymore. To embed a rectangular torus isometrically and piecewise linearly in \( \mathbb{E}^3 \) we take the polyhedron \( F(\mathbb{T}^2) \) defined in section 3 and we intersect it with the plane \( z = 0 \). This plane divides \( F(\mathbb{T}^2) \) in two parts, \( F_1 \) contained in the space \( z > 0 \) and \( F_2 \) contained in the space \( z < 0 \). Let \( \beta > 0 \), we cut \( F(\mathbb{T}^2) \) through this plane and we translate \( F_1 \) vertically by the vector \((0,0,\beta)\) and \( F_2 \) by the vector \((0,0, -\beta)\) with \( \beta > 0 \). We replace the cut triangles \( A_iC_iC_i' \), \( A'_iC'_iA_i, \) \( D_iB_iD_i' \) and \( D'_iB'_iB_i \) by new triangles \( A_{i,\beta}C_{i,\beta}C_{i,\beta}', A'_{i,\beta}C'_{i,\beta}A_{i,\beta}, \) \( D_{i,\beta}B_{i,\beta}D_{i,\beta}' \) and \( D'_{i,\beta}B'_{i,\beta}B_{i,\beta} \) where for \( X \in \{A, A', B, B', C, C', D, D'\} \), \( X_{i,\beta} \) denotes \( X_i = (0,0,\beta) \) or \( X_i = -(0,0,\beta) \) according to whether \( X_i \) belongs to \( F_1 \) or \( F_2 \). Observe that the resulting embedded torus remains flat.

The coordinates of \( X_{i,\beta} \) are the given by:

\[
\begin{align*}
A_{i,\beta} &= \Omega_{A,\beta} + r_4 v \left( \frac{2\pi}{5} \right) \\
B_{i,\beta} &= \Omega_{B,\beta} + r_1 v \left( \frac{2\pi}{5} \right) \\
C_{i,\beta} &= \Omega_{*,\beta} + r_3 v \left( \frac{2\pi}{5} \right) \\
D_{i,\beta} &= \Omega_{*,\beta} + r_2 v \left( \frac{2\pi}{5} \right)
\end{align*}
\]

where \( v(\theta) = (\cos \theta, \sin \theta, 0) \), \( r_i \) are given as in equation 4 and:

\[
\begin{align*}
\Omega_{A,\beta} &= \left( 0, 0, \frac{1 - 0.42\ell}{4} + \beta \right) \\
\Omega_{B,\beta} &= \left( 0, 0, \frac{1 - 3.58\ell}{4} + \beta \right) \\
\Omega_{*,\beta} &= \left( 0, 0, \frac{1 - 0.02\ell}{4} + \beta \right)
\end{align*}
\]

Let \( T_\lambda \) be a rectangular flat torus of sides 1 and \( \lambda = 1 + 4\beta \) with triangulation isomorphic to the square case (see Figure 5), changing the vertex coordinates to:

\[
\begin{align*}
a_0 &= \left( 0, \frac{1 + 0.42\ell + 4\beta}{4} \right), \quad b_0 = \left( 0, \frac{1 - 3.58\ell + 4\beta}{4} \right), \\
c_0 &= \left( \frac{1}{10}, \frac{1 + 0.02\ell + 4\beta}{4} \right), \quad d_0 = \left( \frac{1}{10}, \frac{1 - 0.02\ell + 4\beta}{4} \right)
\end{align*}
\]
\[ a'_0 = \left( 0, \frac{3 - 0.42\ell + 12\beta}{4} \right), \quad b'_0 = \left( 0, \frac{3 + 3.58\ell + 12\beta}{4} \right), \]
\[ c'_0 = \left( \frac{1}{10}, \frac{3 - 0.02\ell + 12\beta}{4} \right), \quad d'_0 = \left( \frac{1}{10}, \frac{3 + 0.02\ell + 12\beta}{4} \right), \]

We define the PL map \( F_\lambda \) in a similar way as in Section 2, \( F_\lambda : T_\lambda \to \mathbb{R}^3 \) as:

\[
F_\lambda(a_i) = \Omega_{A,\beta} + \frac{r_4}{4} v \left( \frac{2\pi}{5} \right) \quad F_\lambda(a'_i) = -\Omega_{A,\beta} + \frac{r_4}{4} v \left( \frac{2\pi}{5} \right)
\]
\[
F_\lambda(b_i) = \Omega_{B,\beta} + \frac{r_1}{4} v \left( \frac{2\pi}{5} \right) \quad F_\lambda(b'_i) = -\Omega_{B,\beta} + \frac{r_1}{4} v \left( \frac{2\pi}{5} \right)
\]
\[
F_\lambda(c_i) = \Omega_{+\beta} + \frac{r_3}{4} v \left( \frac{(2i+1)\pi}{5} \right) \quad F_\lambda(c'_i) = -\Omega_{+\beta} + \frac{r_3}{4} v \left( \frac{(2i+1)\pi}{5} \right)
\]
\[
F_\lambda(d_i) = \Omega_{+\beta} + \frac{r_2}{4} v \left( \frac{(2i+1)\pi}{5} \right) \quad F_\lambda(d'_i) = -\Omega_{+\beta} + \frac{r_2}{4} v \left( \frac{(2i+1)\pi}{5} \right)
\]

To show that this map is isometric it is enough to proof that the length of every edge in the triangulation of \( T_\lambda \) is preserved under \( F_\lambda \). Similar computations as in the proof of the square case show that this is indeed the case.

**Remark.** It is possible to adapt the construction presented in this paper to build PL embeddings with 6 or more corrugations. This obviously increases the number of vertices, edges and faces of the triangulation, see Figure 6 and [8] for more details.

**References**


Figure 5: Triangulation of a flat rectangular torus.

Figure 6: \( PL \) isometric embeddings of the square flat torus with 6, 10 and 30 corrugations.